Parametric Equations, Function Composition and the Chain Rule: A Worksheet

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1 Parametric Equations

We have seen that the **graph of a function** $f(x)$ of one variable consists of a set of points in the xy-plane. These points are the set

$$\{(x, f(x)) : \ x \in \mathbf{D}\}, \qquad \text{where } \mathbf{D} \text{ is the domain of } f(x)$$

The **graph of a function has many properties. For example, every such graph passes the "vertical line test". This test reflects the fact that, for every x value, there is exactly one y value, mainly $f(x)$. We notice that x is always the "input" and y is the "output" that we get by evaluating $f(x)$.**

Example 1.1 ***How is the*** **graph of a function** ***different from a*** **function*?***
Solution: The function describes a way to get a number out for each x value you put in. The function doesn't "live" in the plane. The graph of a function, on the other hand, describes the set of points $\{(x, f(x))\}$ in the xy-plane.

Parametric equations are just another way of describing a set of points in the xy-plane in our case (or in higher dimensions in general). Instead of describing these points by $\{(x, f(x))\}$, we describe the points by the set $\{(x, y)\}$, where x itself (as well as y) is determined by a function $x(t)$ (or $(y(t))$. Here t is the "input", and it is called a "parameter". This is similar to how x is the input in the

case of a graph of a function. Similarly, y is actually a function $y(t)$ dependent on the parameter t. Given a particular value of t, one can find a point in the xy-plane by evaluating $(x(t), y(t))$. Another way that people write parametric equations is

$$x = f(t) \text{ and } y = g(t)$$

for some range of t values. The functions $f(t)$ and $g(t)$ replace the functions $x(t)$ and $y(t)$, respectively, but they are just different names for the same functions.

Example 1.2 *Parametric Equations (Basic)*

One use of parametric equations is that it doesn't rely on the resulting points $\{(x, y)\}$ to actually be a graph of a function. For example, the parametric equations

$$x = 1 \text{ and } y = t, \ \ t \in [0, 4]$$

describes a vertical line segment given by points $\{1, y\}$ where y goes from 0 to 4. This obviously doesn't pass the vertical line test and could not be the graph of a function.

Example 1.3 You Try It

1. Describe the vertical line segment that goes from $(3, 7)$ to $(3, 14)$ using parametric equations.
2. Do problem 21 p. 65

Another important example is the case of a circle of radius r.

Example 1.4 You Try It
Find parametric equations for the circle of radius 5 centered around the origin. If you have trouble, consult the book on page 61.

You can find the formal definition of parametric equations in your text. The "equations" are $x = f(t)$ and $y = g(t)$. They are called "parametric" because they depend on a parameter t.

Remark 1 ***Any particular point*** (x, y) ***on the curve described by parametric equations*** $x = f(t)$ ***and*** $y = g(t)$ ***is obtained by a*** **particular** ***choice of*** t. ***You cannot use one choice of*** t ***for finding*** x ***and a different choice for finding*** y. ***This is illustrated in the next example.***

Example 1.5 **The motion of a fly is described by the equations**

$$x = -\cos(t) \textbf{ and}$$
$$y = \sin(t),\ \ t \in [0, 2\pi]$$

At what time is the fly at the position $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$**?**

Solution: "At what time" means we're looking for a value of t **that gives us the point** $(x, y) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. **This expression**

$$(x, y) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$$

is really two equations that we need to solve using the fact that $x = \cos(t)$ **and** $y = \sin(t)$, **mainly**

$$-\cos(t) = \frac{\sqrt{2}}{2}$$

and

$$\sin(t) = \frac{\sqrt{2}}{2}.$$

However we need to solve these equations ***simultaneously***, **i.e. the fly has to be in the appropriate** x **position** ***and*** **in the appropriate** y **position** ***at the same time.***

In the first equation, we find that $t = \frac{3\pi}{4}$ **and** $t = \frac{5\pi}{4}$ **are two solutions to the equation. For the second equation, we find that** $t = \frac{\pi}{4}$ **and** $t = \frac{3\pi}{4}$ **are both solutions. The only** ***simultaneous*** **solution is** $t = \frac{3\pi}{4}$, **when both equations are satisfied for the same** t **value.**

2 Composition of Functions and Parametric Equations

Recall that if $f(x)$ and $g(x)$ are two functions and the range of $g(x)$ is in the domain of $f(x)$, then we can form the composition

$$f(g(x)).$$

The goal of this section is to understand this composition of functions better.

Example 2.1 ***This example illustrates the calculations involved with composition.***

Find the composition $f(g(x))$ when $f(x) = \sin x^3$ and $g(x) = \frac{1}{x^2}$.
Solution: First make sure you're clear on the notation: $\sin x^3 = \sin(x^3)$ which is NOT the same as $\sin^3 x = (\sin x)^3$.

$$f(g(x)) = \sin\left(g(x)^3\right) = \sin\left[\left(\frac{1}{x^2}\right)^3\right] = \sin\left(\frac{1}{x^6}\right)$$

or alternatively

$$f(g(x)) = f\left(\frac{1}{x^2}\right) = \sin\left[\left(\frac{1}{x^2}\right)^3\right] = \sin\left(\frac{1}{x^6}\right).$$

Example 2.2 You Try It

1. **Do Problems 37, 39 on p. 22.**

Example 2.3 ***The profit made on orange juice as a function of volume. This example illustrates the concept of composition.***

Imagine that x is an amount (volume) of orange juice in litres. Let $g(x)$ be the price of buying x litres of orange juice. Suppose that $g(x) = 3x$, so it costs \$3 per litre of orange juice. Now suppose that $f(x)$ is the amount of money the company earns when collecting \$$x$. For example, $f(x) = .2x$, i.e. the company has a profit of 20 cents for each dollar collected. Notice that x does NOT stand for the same thing in the context of $f(x)$ and in $g(x)$. As a "variable" in

the domain of $g(x)$, x is an amount of orange juice. As a variable in the domain of $f(x)$, x is a quantity of money. The important thing, however, is that ***the range of*** $g(x)$ ***is the domain of*** $f(x)$***; both are measured in dollars.*** Now what does $f(g(x))$ mean? Since x is first taken in by the function $g(x)$ (on the inside), we know that x must stand for an amount of orange juice. Now $g(x)$ is the price of that orange juice, and $f(g(x))$ is the amount of profit taken in for that price. Thus $f(g(x))$ represents the amount of profit obtained when x litres of orange juice are sold.

Here's the explicit calculation:

$$f(g(x)) = .2g(x) = .2(3x) = 1.5x$$

or alternatively,

$$f(g(x)) = f(3x) = .2(3x) = 1.5x.$$

Now let's do a parametric composition.

Example 2.4 Suppose that an ant moves along the graph of the parametric equations given by

$$x = 2\cos t \textbf{ and} y = 3\sin t \textbf{ where } t \in [0, 2\pi).$$

at any time t in the domain. First, convince yourself that this is an ellipse by finding the Cartesian equation that these parametric equations satisfy. Notice that $3x = 6\cos t$ and $2y = 6\sin t$, which is like a circle of radius 6. From this, you might guess that $(3x)^2 + (2y)^2 = 36$. Dividing both sides by 36, you'll find the equation of an ellipse in standard form.

Now suppose that the position of a bird depends on the position of the ant. Suppose that the bird can be found at the position $x_{bird} = 2x_{ant}$ and $y_{bird} = 5y_{ant}$. Can you figure out the position of the bird as a function of time?

Since the bird's position depends on the ant's position, which in turn depends on time, one suspects this is a composition question, as compositions always reflect a "chain" of dependencies. In this case, we see that $x_{bird} = 2x_{ant} = 2(2\cos t) = 4\cos t$. Similarly, $y_{bird} = 2y_{ant} = 2(3\sin t) = 6\sin t$. Thus the bird is also on a (different) ellipse in the xy-plane.

Here is a picture of what's going on when finding $f(g(x))$. Take x and first apply g to it, to get $g(x)$:

$$x \longrightarrow g(x)$$

Now apply f to the value $g(x)$, to get $f(g(x))$:

$$g(x) \longrightarrow f(g(x)).$$

Example 2.5 ***Recognizing functions as compositions.*** **How do you recognize a function as a composition of others? There may be more than one answer!!**

1. **If you're given $f(x) = x^7 \sin x^2$, this is NOT the composition of x^7 and $\sin x^2$, but the product of these two functions. However, here is one way of seeing a composition in here: $\sin x^2$ is a composition. Let $g(x) = x^2$, and $h(x) = \sin x$. Then $h(g(x)) = \sin g(x) = \sin x^2$. Notice that $g(h(x)) = (h(x))^2 = (\sin x)^2 = \sin^2 x$, which is not the same as $\sin x^2$, so the order of composition if important.**

2. **Again, let $f(x) = x^7 \sin x^2$. Let $g(x) = x^2$ and $h(x) = x^{\frac{7}{2}} \sin x$. Then**

$$h(g(x)) = (g(x))^{\frac{7}{2}} \sin g(x) = (x^2)^{\frac{7}{2}} \sin x^2 = x^7 \sin x^2 = f(x).$$

   so indeed we can see $f(x)$ as a composition. However, this is not very useful from the point of view of differentiation, since $h(x)$ is hard to differentiate.

3. **If you're given $f(x) = (2x+1)^2$, then the function on the "inside" is $g(x) = 2x + 1$. What is being "done" to $g(x)$? It's being squared. What function takes anything and squares it? $h(x) = x^2$. Thus $h(g(x)) = (2x+1)^2 = f(x)$.**

4. **$f(t) = (1 + \cos 2t)^{-4}$. The "inside" is $g(t) = 1 + \cos 2t$. The outside function takes whatever is on the inside to the -4th power. Thus $h(x) = x^{-4}$. Then**

$$h(g(t)) = (g(t))^{-4} = (2\cos 2t)^{-4}.$$

Notice that it didn't matter that we wrote $h(x)$ as a function of x. Since x is just a variable, it stands for "whatever you feed into h".

Notice also in this case that $g(t)$ is also a composition of functions. If we let $k(t) = 2t$ and $m(t) = 1 + \cos t$, then $g(t) = m(k(t)) = 1 + \cos(k(t)) = 1 + \cos 2t$.

5. $f(t) = (1 + \cos 2t)^{-4}$ (again). Notice that you could have broken the function up differently to begin with. Let $g(t) = 2t$. Let $h(x) = (1 + \cos x)^{-4}$. Then

$$h(g(t)) = h(2t) = (1 + \cos 2t)^{-4} = f(t).$$

This way of breaking up the equation is less useful from the point of view of differentiating functions, since $h(x)$ is hard to differentiate without using the chain rule again. We'll see more on the chain rule below.

Example 2.6 You Try It

1. Do Exercise 41 on p. 22

3 Taking the Derivative of the Composition of Functions: the Chain Rule

Theorem 3.1 (The Chain Rule) *The chain rule specifies how to differentiate a composition of functions. Let $f(x) = h(g(x))$. Then*

$$f'(x) = h'(g(x))g'(x).$$

How do you calculate $h'(g(x))$?

1. Write down the function $h(x)$
2. Differentiate $h(x)$ with respect to x (don't worry about $g(x)$).
3. Evaluate $h'(x)$ at $g(x)$.

Example 3.1 Consider the function $f(x) = (1+\cos x)^{-4}$. We notice that the "inside" is $g(x) = 1+\cos x$, and the outside function is $h(x) = x^{-4}$. Then

$$h(g(x)) = (g(x))^{-4} = (1+\cos x)^{-4} = f(x)$$

so we may apply the chain rule. We first calculate $g'(x)$:

$$g'(x) = 0 + -\sin x = -\sin x.$$

Now we calculate $h'(g(x))$ using the steps outlined above.

1. **Write down $h(x)$: $h(x) = x^{-4}$.**
2. **Differentiate $h(x)$ with respect to x: $h'(x) = -4x^{-5}$, using the power rule.**
3. **Evaluate $h'(x)$ at $g(x)$: $h'(g(x)) = -4(g(x))^{-5} = -4(1+\cos x)^{-5}$.**

Lastly we use the chain rule:

$$f'(x) = h'(g(x))g'(x) = -4(1+\cos x)^{-5}(-\sin x) = +4(\sin x)(1+\cos x)^{-5}$$

Notice that the final answer is just using the power rule to $(1+\cos x)^{-4}$ times the "derivative of the inside", which is $-\sin x$.

Example 3.2 You Try It

1. **Express $f(x) = \sin(2x^2-1)$ as the composition of two functions. What is $g(x)$ and what is $h(x)$? Check that $h(g(x)) = f(x)$.**
2. **Find $f'(x)$ using the chain rule.**
3. **Do Exercises 7,9,12,13,14,17,19, 23, 27, 30 on page 195 using the techniques outlined above. Check your answer in the back of the book.**

4 Another look at the chain rule

You can also do the chain rule by applying the formula:

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}.$$

Why is this the same as above? The answer is very tricky!! Part of the trick is just notational. Let $u = g(x)$. Then $f(x) = h(g(x)) = h(u)$. The original chain rule says:

$$\frac{df}{dx} = h'(g(x))g'(x).$$

The term $g'(x)$ is clearly the same as $\frac{dg}{dx} = \frac{du}{dx}$ since $u = g(x)$. Let's take a closer look at $h'(g(x))$. Here's the first trick: The functions $h(x)$ and $h(u)$ are really the ***same thing*** except there's a variable change. Similarly, $h'(x) = \frac{dh}{dx}$ is describing the ***same function*** as $h'(u) = \frac{dh}{du}$, except the variable u is used instead of x. Then $h'(g(x))$ (where the derivative is done with respect to x, and then $g(x)$ is stuck in for x, is the same as $\frac{dh}{du}$, take the derivative of h as a function of u and then evaluates at $g(x)$. However, $u = g(x)$ means that you could can just consider $\frac{dh}{du}$ already as a function of u and you don't need to stick in $g(x)$. In practice, in order to evaluate $\frac{dh}{du}$, we do actually plug in $g(x)$ for u, since we want it as a function of x in the end. This would lead us to the conclusion that

$$\frac{df}{dx} = \frac{dh}{du}\frac{du}{dx}.$$

So here's the second trick: If we consider f as a function of u, we mean that $f = h(u)$! This is a subtle notational issue. To illustrate this point, consider $f(x) = ((x^2) + 1)^3$. On the one hand, when we write $f(x)$ to mean "stick x into the formula". In this light, if we write $f(u)$, we would mean $f(u) = ((x^2) + 1)^3$. But considering f as a function of u is different: let $u = x^2 + 1$. Then by saying "f as a function of u," we really mean the function u^3. This is exactly the "outside function", or what we have been calling $h(u)$!! Therefore, $\frac{dh}{du} = \frac{df}{du}$, and sticking that into the equation above, we obtain

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}.$$

The main point in working problems using this, is that $\frac{df}{du}$ means to think of f as a function of u, where u is a convenient substitution that makes f look simpler. It is not actually $f(u)$, but $h(u)$, the outside function in our original discussion.

Example 4.1 Differentiate $\cos(t^3 - 5t)$ using the second form of the chain rule.

Solution: We use the formula (with t replacing x):

$$\frac{df}{dt} = \frac{df}{du}\frac{du}{dt}.$$

Let $u(t) = t^3 - 5t$. Then $f = \cos u$ and $\frac{df}{du} = -\sin u = -\sin(t^3 - 5t)$. Also, $\frac{du}{dt} = 3t^2 - 5$. The final answer is then

$$\frac{df}{dt} = \frac{df}{du}\frac{du}{dt} = -\sin(t^3 - 5t)(3t^2 - 5) = -(3t^2 - 5)\sin(t^3 - 5t).$$

5 Chain Rule and Parameterized Curves

You may be wondering: what does the chain rule and differentiating functions have to do with parametric curves? The main answer is the following: if you are given a curve in parametric equations, how do you find the tangent line to a curve? In particular, how do you find its slope?

Remark 2 ***If you have a curve described by the*** **graph of a function** $f(x)$***, then the slope of the tangent line at a point*** $P(x_0, y_0)$ ***is*** $f'(x_0)$ ***(the derivative*** $f'(x)$***, evaluated at*** P***).***

Remark 3 ***If you have a curve described by*** **parametric equations** $x = f(t)$ ***and*** $y = g(t)$***, then the slope of the tangent line at a point*** $P(x_0, y_0)$ ***is given by*** $\frac{dy}{dx}$***, evaluated at*** P***.***

So the real question boils down to: How do you calculate $\frac{dy}{dx}$ for a parametric curve (a curve described by parametric equations)?

The answer is using the CHAIN RULE for parametric equations

Theorem 5.1 (The Chain Rule for Parametric Curves) ***Let*** $x = f(t)$ ***and*** $y = g(t)$ ***be parametric equations for a curve, where*** $t \in D$***,*** D ***a domain in*** $\mathbb{R}$***. Then***

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{dg/dt}{df/dt}.$$

You may wonder: How is this the chain rule? Multiply both sides of the equation by dx/dt **and get**

$$\frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dt}$$

which is the chain rule for calculating $\frac{dy}{dt}$**. This seems weird: we** ***have*** y **as a function of** t **(mainly** $g(t)$**). So what does the left hand side of the equation mean? We are just applying a trick here. We'd LIKE to have** $\frac{dy}{dx}$ **even though we have no explicit expression for** y **in terms of** x**, as we do with Cartesian functions of the form** $y = f(x)$**. So we PRETEND that** y **were a composition with** $x(t)$**. If it were, i.e.** $y(t) = h(x(t))$**, then by the same reasoning above, we'd obtain that**

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

where x **is playing the role that** u **does in the previous section. Now it happens that we know** $\frac{dy}{dt}$ **and** $\frac{dx}{dt}$**, so we may solve for** $\frac{dy}{dx}$ **and it doesn't matter that we never actually wrote** y **down as a composition of functions!!**

Let's try to use this formula:

Example 5.1 Let $x = \sin t$**,** $y = \sqrt{5}\sin t$**. Find the slope of the tangent curve at** $t = 2\pi/3$**. Then find the equation of the line through this point tangent to the curve.**
Solution: We use the formula:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

By direct calculation, $\frac{dy}{dt} = \sqrt{5}\cos t$ **and** $\frac{dx}{dt} = \cos t$**. The ratio is then**

$$\frac{dy}{dx} = \sqrt{5}$$

We evaluate at $t = 2\pi/3$, but there is no t-dependence, so the answer is $\sqrt{5}$. To find the line through the curve, we use

$$y = mx + b = \sqrt{5}x + b.$$

Using the point $(\sin 2\pi/3, \sqrt{5}\sin 2\pi/3) = (\frac{\sqrt{3}}{2}, \frac{\sqrt{15}}{2})$ we solve for b:

$$\frac{\sqrt{15}}{2} = \sqrt{5}\frac{\sqrt{3}}{2} + b$$

so $b = 0$ and the equation of the line is just $y = \sqrt{5}x$.

Example 5.2 You Try It:

1. **Let $x = \cos^2 t$ and $y = \sin t$, $t \in [0, 2\pi)$. Find the slope of the tangent line to the curve at $t = \frac{\pi}{4}$ by finding**

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

   and then evaluating at $t = \frac{\pi}{4}$. Then find the equation of the line tangent to the curve at this point.

2. **Do problems 33, 35, 37, 39 on p. 195.**