

CALCULATION ACROSS CULTURES AND HISTORY

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ABSTRACT. This paper describes a series of activities for both students and teachers that demonstrate various methods of calculation. Included are examples using Napier's bones and other calculating rods, slide rules, abacuses, Vedic mathematics, and a variety of paper and pencil techniques. The activities are designed to provide an opportunity to discuss culture, geography, history, and mathematics. They have been used successfully with elementary school children and their teachers as part of an ongoing program sponsored by the K-12 International Education Outreach at the International Cultural Center of Texas Tech University. In addition they have been used with both social studies and mathematics high school teachers.

1. INTRODUCTION

The work described in this paper has grown out of a project for elementary school students created by the Department of Mathematics and Statistics at Texas Tech University and the K-12 International Education Outreach at the International Cultural Center of Texas Tech University. Every semester the project gives a two and half hour workshop for up to 60 fourth through sixth grade students and their teachers. The workshop is staffed by professors, personnel from the International Cultural Center, and both undergraduate and graduate mathematics students from the student chapters of the Mathematical Association of America and the Society for Industrial and Applied Mathematics. The purpose is to expose the elementary school students to some interesting examples of calculation that permit the introduction of elements of cultural diversity, geography, history, and mathematics. The activities include not only calculation but also coloring of maps, identifying capitals of countries, and discussions of various historical dates. For an online presentation describing this workshop see [24].

In this paper we describe some of the calculation activities that we have used in the workshop. We do not attempt to give a detailed or authoritative account of the mathematical, historical, or cultural basis of these methods of calculation. We do, however, attempt to provide the beginnings of a set of references, much of them online, for those who would like to delve into the subject in more depth or to duplicate this outreach program. The need for such outreach by mathematicians is often taken for granted; however, once one has seen an elementary school teacher coloring in Cuba because she thought it was Alaska and thus part of the United States, it becomes obvious that our job as educators is even more challenging than

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$$\begin{array}{r}
 27 \\
 \cancel{3}1\cancel{8}13 \\
 - 184 \\
 \hline
 199
 \end{array}
 \qquad
 \begin{array}{r}
 3183 \\
 - \underset{1}{1}\underset{1}{1}84 \\
 \hline
 199
 \end{array}$$

FIGURE 1. The calculations on the left are those normally followed by students performing the subtraction $383 - 184$ in the United States. Those on the right are the calculations normally followed by students in Europe.

just addressing mathematics. We also believe that there is much important work that still needs to be in the area of documenting calculation techniques and describing the history of these techniques. Because our intended audience here is not mathematical historians, we give a vastly simplified discussion of the historical topics. For a scholarly review of a recent book criticized for adopting an oversimplified view see [7]. For an online discussion list of mathematical history issues see the math-history-list sponsored by the Math Forum of Drexel University [32].

2. PAPER AND PENCIL

Most of us are taught the basics of numerical calculation before the age of 10. Thus calculation is very fundamental to us and in fact most of us when asked to perform a calculation will immediately revert to our native language when performing the task. In this activity we discuss various ways to calculate using paper and pencil including algorithms for subtracting, dividing, and multiplying. Specifically, we include a discussion of a simple way to keep track of borrowing when subtracting that is taught in Europe. We also discuss how this method of subtraction can be generalized and lead to a method taught to European (and South American) elementary school students for handling long division. Finally we describe three ways to multiply: the lattice method that is sometimes taught in elementary schools in the United States, an ancient Egyptian algorithm, and the similar Russian peasant's algorithm.

2.1. Subtraction. In the United States most elementary aged students learn how to borrow when subtracting as follows: if a digit in the subtrahend is greater than the corresponding digit in the minuend, then 10 is added to the smaller digit before doing the subtraction and 1 is borrowed (or subtracted) from the digit of the minuend immediately to the left. See Figure 1 on the left.

European elementary school students are instead taught that rather than subtract 1 from the digit of the minuend, they should add 1 to the corresponding digit

$$\begin{array}{r}
 2587 \\
 38 \overline{) 98306} \\
 \underline{76} \\
 223 \\
 \underline{190} \\
 330 \\
 \underline{304} \\
 266 \\
 \underline{266} \\
 0
 \end{array}
 \qquad
 \begin{array}{r}
 98306 \mid 38 \\
 223 \mid \underline{2587} \\
 330 \mid \\
 266 \mid \\
 0 \mid
 \end{array}$$

FIGURE 2. The calculations on the left are those normally followed by students performing the long division 98306 by 38 in the United States. Those on the right show what many students in Europe and South America would write to perform the same division.

of the subtrahend. See Figure 1 on the right for a detailed example. In this example since 4 is greater than 3 a small 1 is written just to the left of the 3 in the minuend while another small 1 is written just to the left of the 8 in the subtrahend. Computing $13-4=9$ we write a 9 in the answer in the units place. Adding the small one and the 8 together to get 9 we see that 9 is greater than 8 and a small 1 is written just to the left of the 8 in the minuend and another 1 just to the left of the 1 in the subtrahend. Computing $18-9=9$ we write a 9 for the answer in the 10's place. Adding the small 1 to the 1 in the subtrahend to get 2 and we compute $3-2=1$, which we write in the 100's place in the answer. As students become more comfortable with this method they typically write down only one of the 1's for each borrow. For an interesting description of this algorithm and why it is less confusing for students to use see [12]. In the following subsection we see how this method can be generalized to enable a much more compact way of doing long division than that which is typically taught in the United States.

2.2. Long Division. In the United States many students are taught to manage long division as shown in Figure 2 on the left. Notice that in the example each multiplication and each difference occupies a different horizontal line and thus there is much writing and indeed the division is *long*. Figure 2 on the right shows what many students in Europe and South America would write down to perform the same division. When shown this difference most people that were taught the first method are amazed that elementary students can be taught to perform a multiplication and subtraction simultaneously and obtain the required results. The techniques used to manage subtraction described in the preceding paragraph can be used here.

$$\begin{array}{r|l}
 98306 & 38 \\
 \hline
 223 & 2587 \\
 330 & \\
 266 & \\
 0 &
 \end{array}$$

FIGURE 3. These are the calculations normally followed by students performing the division 98306 by 38 in Europe or South America along with annotations to aid in the calculations.

Figure 3 shows the same calculations as in Figure 2 on the right along with some helpful annotations that we will explain. First we determine that 38 will go into 98 a little over 2 times. A 2 is written down on the answer line and $2 \times 8 = 16$ is computed mentally. Since 16 can not be subtracted from 8 we write down a small 1 just to the left of the 8. (Essentially we are borrowing a 1.) Now $18 - 16 = 2$ so we write down a 2 under the 8. Now $2 \times 3 + 1 = 7$ so we write down 2 because $9 - 7 = 2$. (The 3 came from the three in 38 and the 1 was what we had borrowed). Bringing down the 3 (from 98306) we are now dividing 223 by 38. Since 38 will go into 223 a little over 5 times, we write down a 5 to the right of the 2 on the answer line. Then $5 \times 8 = 40$ is computed mentally. Since 40 can not be subtracted from 3 we write down a small 4 to the left of the 3. (Here we borrow 4.) Computing $43 - 40 = 3$ we write down 3 under the 3. Computing $5 \times 3 + 4 = 19$, which is subtracted from 22, we write down a 3 under the 22. After bringing down the 0 we are dividing 38 into 330, which results in an 8. We write down the 8 on the answer line. But $8 \times 8 = 64$ is too big to subtract from 0 so we write down a little 7 to the left of the 0. Computing $70 - 64 = 6$ we write down a 6 under the 0. Now we compute $8 \times 3 + 7 = 31$, which is subtracted from 33 to obtain 2. Bringing down the 6 from above we are now dividing 266 by 38, which is exactly 7. Although this method is tedious to explain in a written document, it can usually be shown to a 6th grader who is walked through several examples in about fifteen to twenty minutes.

During this activity we talk about where Europe and South America are. We mention that in South America the predominant language is Spanish with the exception of Brazil where Portuguese is spoken. In Europe, Spanish is spoken in Spain. We point out where France is within Europe. The students will also discuss France during the activity on calculating rods.

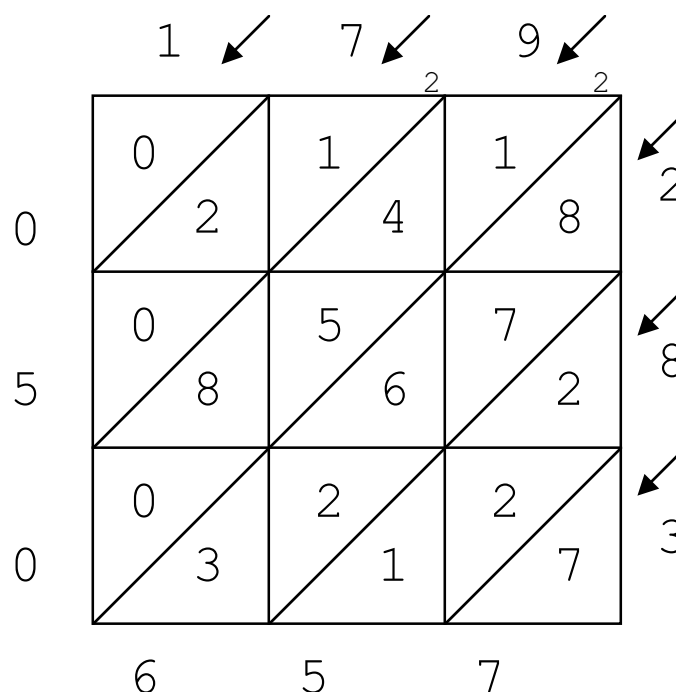


FIGURE 4. The lattice method being used to compute $179 \times 283 = 050,657$

2.3. Lattice Multiplication. Lattice multiplication (also known as the gelosia or jalousie method) was probably invented by Hindu mathematicians sometime in the first millennium. The Persian mathematician al-Karaji (also al-Karkhi) (953-1029) who was most likely from either the present day Iraqi city, Baghdad, or from the Iranian city, Karaj, described the method in his book, *Kafi fil Hisab*, written in 1010 [23]. Lattice multiplication was first introduced in Europe by Leonardo Pisano Fibonacci (1170-1250) [25]. Fibonacci is also known for the sequence

$$1, 1, 2, 3, 5, 8, 13, \dots, F_{n+2} = F_{n+1} + F_n, \dots$$

that he used to describe how a population of rabbits grow [22]. To see how lattice multiplication can be used to compute $179 \times 283 = 50,657$ see Figure 4. The numbers to be multiplied are found along the top and right side of the big box. The product of the single digits is shown in each of the smaller boxes of the lattice. The answer is read around the left and bottom sides of the big box. These digits of the answer are the sum of the digits in the lattice down along the diagonals indicated by the arrows. Notice the two small 2's that denote the carries. Later the Scottish mathematician, John Napier, was inspired by this method to create the calculating rods that became known as Napier's bones. Lattice multiplication is still taught in some elementary schools and therefore familiar to some of the students in the workshop.

1	39*	45-32=13	39
2	78	13-8=5	156
4	156*	5-4=1	312
8	312*		<u>+1248</u>
16	624		
32	1248*		45x39 = 1755

FIGURE 5. Multiplying 45×39 using an ancient Egyptian algorithm based on doubling.

The activity on lattice multiplication permits the introduction of India, Iraq, and Iran. Iraq, and its capital, Baghdad, are of special interest to the students because of its almost constant mention in current news. The students will get another chance in the activity on Vedic mathematics to discuss India. Ancient Babylonia (Iraq) is also mentioned in the activity on the abacus. During the time periods mentioned in the activity Europe was in the middle of the Medieval period of history.

2.4. Egyptian Multiplication and the Russian Peasant's Algorithm. The Rhind papyrus is an ancient Egyptian document dating from 1650BCE written by Ahmes, a scribe, who in the document claims he is copying a document from around 2000BCE [21]. The papyrus was “found” by Alexander H. Rhind, an egyptologist, who purchased it in 1858. It is one of the oldest mathematical writings. It describes a way to multiply based on doubling. Consider the product 45×39 , see Figure 5. On the left of the figure we show two columns of numbers starting at the top with 1 and 39. Each subsequent row is the double of the row above it. We continue this until the double of the left most number is greater than 45. Note that the left most column is just the powers of 2. In the middle of the figure we show subtracting the largest power of two that is possible from 45 and then from $45-32=13$ and so on. We mark each row with an asterisk if we subtracted the corresponding power of 2. On the right side of the figure we add the doublings of 39 which we marked. The sum is 1755, which is the product of 45×39 . Students that have played with binary representations will immediately see the relationship to this algorithm.

Another algorithm based on doubling is the Russian peasant's algorithm. Here again two columns of numbers are formed under each of the two multiplicands. Each row is formed by dividing the first number by 2 and discarding the remainder and doubling the second number. See Figure 6. Now each row that has an odd number in the first column is marked with an asterisk. The numbers in the second column that are in a marked row are added to form the product.

This activity allows us to talk about ancient Egypt. The ancient Egyptian pyramids were built during the old and middle kingdoms. The three most famous of these date from the beginning of the old kingdom in Giza near Cairo circa 2500BCE. The Rhind papyrus dates from over a 1000 years later at the end of the middle kingdom [4]. The Russian peasant algorithm provides an opportunity to talk about Russia, the largest (in terms of land mass) country in the world. Russia is also mentioned in the activity on the abacus.

3. CALCULATING RODS

This activity describes various approaches to using rods to calculate. Calculating with rods is sometimes called *rabdologia*, which comes from the Greek words for *rod* and *word*. We describe how to multiply with two kinds of rods, one called *Napier's bones* and the other called *Genaille-Lucas rods*. Although we only give examples of how to multiply in this paper; it is possible to do more complicated operations, for example, division and extracting square and cube roots [29, 2]. To obtain a copy of several of the rods in .pdf format that are suitable for reproducing on cardboard see [3].

3.1. Napier's Bones. John Napier (1550-1617) was a Scottish nobleman who made contributions to both mathematics and theology. John Napier is best known now as the inventor of logarithms (1614) and natural logarithms are still sometimes referred to as Napierian logarithms. At the time of Napier's life, it was thought by some neighbors that because of his amazing intelligence he was in league with the devil. In 1617 just before he died John Napier invented a way of doing calculation with rods, which became to be known as Napier's bones because they were originally made of bone. These rods were based on the lattice method of multiplication. The multiplication rods consist of a rod for each digit 0 through 9 along with an index rod. Figure 7 shows how they can be used to calculate 7×286 . Notice that we must do the addition and keep track of the carriers when using Napier's bones.

When we work on this activity we point out that Scotland is north of England and is a part of Great Britain, which is an island in western Europe on the other side of the Atlantic ocean from the United States. Although most elementary students with which we interact do not have any idea where (or what) Scotland is, many can locate on a world map the United States, the Atlantic ocean, and Europe.

To help students relate the dates mentioned to something they might know we recall that Columbus landed in America in 1492, that Shakespeare (1564-1616) was a contemporary of John Napier, and that the Mayflower sailed from England for the United States in 1620.

3.2. Genaille-Lucas Rods. Édouard Lucas (1842-1891) was a French mathematician who worked in number theory and finding large primes. He maybe is best known to non-mathematicians for inventing the Tower of Hanoi puzzle where disks stacked in a conical pile on one peg are moved among this and two other pegs so that a larger disk is never placed over a smaller until all the disks have been moved

$$\begin{array}{r}
 45 \quad 39^* \\
 22 \quad 78 \\
 11 \quad 156^* \\
 5 \quad 312^* \\
 2 \quad 624 \\
 1 \quad \underline{1248^*} \\
 1755
 \end{array}$$

FIGURE 6. Multiplying 45×39 using Russian peasant algorithm.

to a new peg. In addition the sequence,

$$2, 1, 3, 4, 7, 11, 18, \dots, L_{n+2} = L_n + L_{n+1}, \dots,$$

is called the Lucas sequence. This sequence, which is similar to the Fibonacci sequence, is also sometimes known to elementary school students.

Because of the difficulty with handling carries when using Napier's bones Édouard Lucas posed the problem of inventing a set of rods that did not require performing additions. The French civil engineer, Henri Genaille, solved the problem and by 1888 [28] he and Lucas were writing about the solution. See Figure 8 to see how to multiply 7×286 using Genaille-Lucas rods. In this example one needs to know that 7 times 6 ends in a 2 and then it is just necessary to follow from the right starting

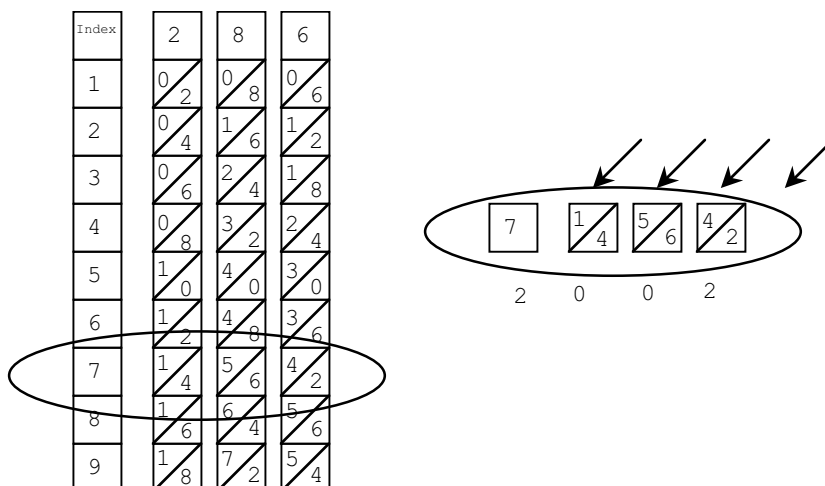


FIGURE 7. Multiplying $7 \times 286 = 2002$ using Napier's bones.

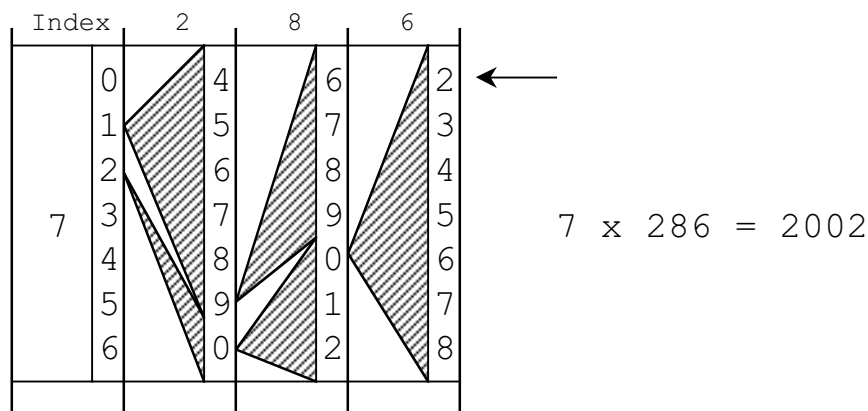


FIGURE 8. Multiplying $7 \times 286 = 2002$ using Genaille-Lucas rods. Only a part of the rods is shown.

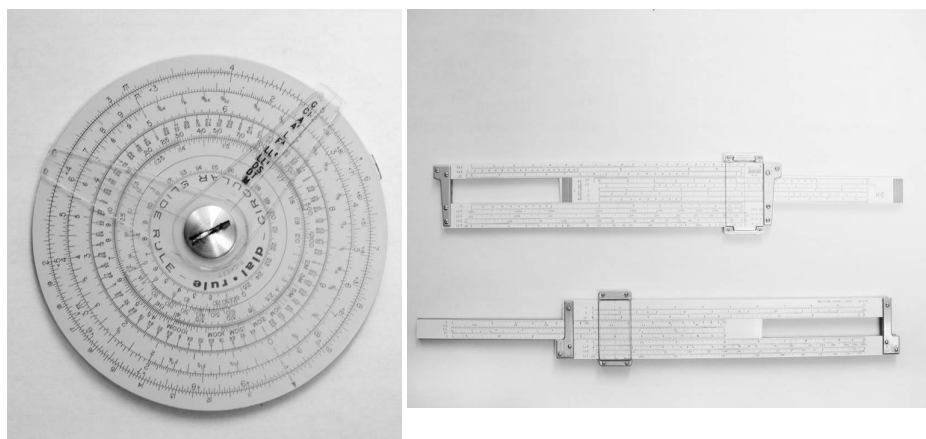


FIGURE 9. A circular and rectilinear slide rule.

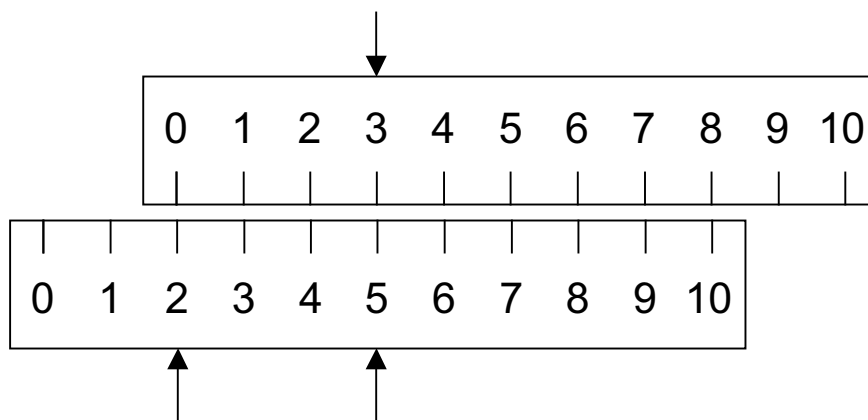
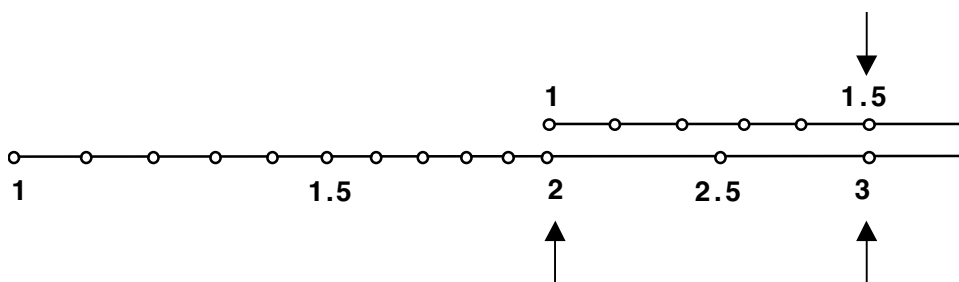
with the 2 going to the left reading off the digits that are pointed to by the darkened triangles. Thus the triangle with 2 on its right edge points to 0, which is on the right edge of a triangle that points to 0, which is on the right edge of a triangle that points to 2 and the number is read off from the least to most significant digit. Genaille and Lucas also invented rods to do division. See [3] for reproductions of Genaille-Lucas rods for both multiplication and division.

For the social studies part of this activity students are shown France on the map along with its capital, Paris. To help the students relate the date 1888 to what they already know, the Statue of Liberty can be discussed. This great statue on Liberty Island in New York was designed and completed in 1886 by the French sculptor, Frederic Bartholdi (1834-1904), and was gift from France to the United States [16].

4. SLIDE RULES

The invention of logarithms by John Napier in 1614 made possible the invention of the slide rule. It should be noted that the Swiss mathematician, Jobst Bürgi (1552-1632), is often cited as a simultaneous but independent inventor of logarithms. By 1620 Edmond Gunter (1581-1626) had invented the logarithmic scale; that is, a scale where the numbers 1 through 10 are arranged along a line with their distances from the left edge being determined by their logarithms. By measuring distances along this scale with a set of dividers (or compass) the user could add logarithms and therefore multiply. Soon after this, certainly by 1627, William Oughtred (1574-1660), an English mathematician, and Edmund Wingate (1593-1656) had put a logarithmic scale on each of two rulers and so invented the rectilinear slide rule. The circular slide rule [6] was also invented in this period. See Figure 9 for a picture of both a circular and a rectilinear slide rule.

Because slide rules were so important to engineering before the invention of the hand held calculator in 1972 and the important role of the United States in engineering over the last two centuries, we talk about the United States, and its capital, Washington, during this activity. An engineer that is mentioned here is John A. Roebling (1806-1869), the builder of the Brooklyn bridge [14]. We also discuss Benjamin Banneker (1731-1806), a mathematician and astronomer, who worked

FIGURE 10. Computing $2 + 3 = 5$ on adding slide rules.FIGURE 11. Computing $2 \times 1.5 = 3$ on multiplying slide rules.

with Pierre L'Enfant to survey and plan Washington, DC. Benjamin Banneker is sometimes referred to as the first African-American mathematician [5].

To introduce students to slide rules they are first shown how to add with *adding slide rules*, which simply consist of two identical rulers each with a linear scale. Figure 10 shows the addition of 2 and 3 on adding slide rules. Then the students are shown *multiplying slide rules*, which consist of two identical rulers each with a logarithmic scale. See Figure 11 to see how the product of two numbers is found using a slide rule. Through the generosity of several of our faculty the students can then work with several actual slide rules. When time permits, division is also explained.

5. ABACUSES

It is believed that the abacus was derived from counting boards that were used by several ancient civilizations. Counting boards are a method of arranging and moving markers, e.g., beads or pebbles, on a series of grooves on a tablet or even on lines drawn in the sand. It is believed that these devices were used for calculation in ancient Babylonia as early as 4500 years ago [20] and in China as early as 3000 years ago. Because of the ephemeral nature of the materials from which counting

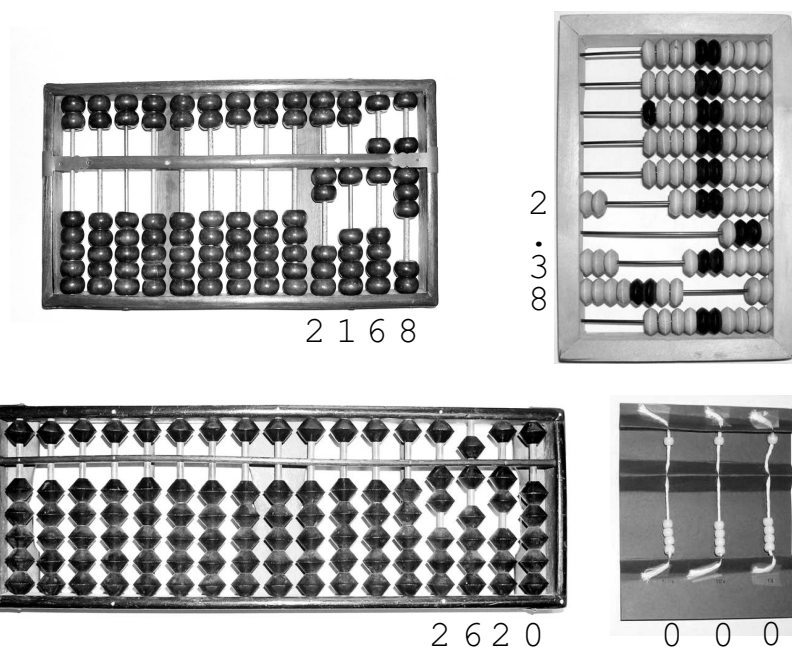


FIGURE 12. The Chinese 2,5 abacus (top left), the Russian abacus (top right), the Asian 1,5 abacus (bottom left), and a student made Japanese 1,4 abacus (bottom right).

boards are made, the earliest known example is the Salamis tablet that dates from a much later time, around 300BCE [8].

In general an abacus is a frame of a set of parallel rods on which digits of a number are represented by sliding beads. We show the students four kinds of abacuses: (1) the Chinese (suan pan - 2,5 abacus); (2) the Russian (schoty); (3) the 1,5 abacus; and (4) the modern Japanese abacus (soroban - 1,4 abacus). See Figure 12. After showing the students the suan pan, which is easier to use, but slower for an expert, we show the students how to use the modern soroban and give them the materials that they can use to make their own abacus at home. It should probably be noted that for addition an expert user of anyone of these abacuses is faster than someone using a modern calculator because the sum develops as the user enters the numbers on the abacus.

Historical references to uses of the suan pan in China occur as early as 1200 CE. This abacus is held so that the rods run vertically. The beads on the rods are divided into two ranks. The top rank contains two beads for every place each representing 5 units. The bottom rank contains 5 beads for every place each representing 1 unit. Thus this abacus is often referred to as the 2,5 abacus. The configuration of the beads in this abacus result in multiple ways of representing the numbers; for example the number 10 can be represented in 3 different ways, see Figure 13. This multiplicity of ways to represent a number makes handling carries when adding easier. The 2,5 abacus was modified to the 1,5 abacus by the Japanese via Korea in the 17th century. Later in the early 20th century (circa 1930) the Japanese again

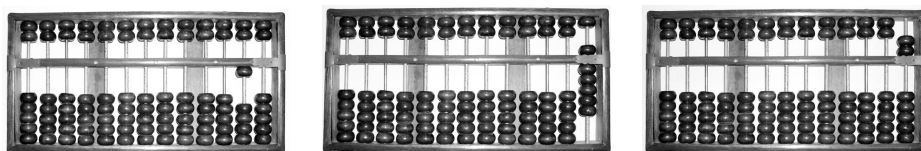


FIGURE 13. The above pictures show three different ways of representing 10 on the suan pan, the Chinese abacus.

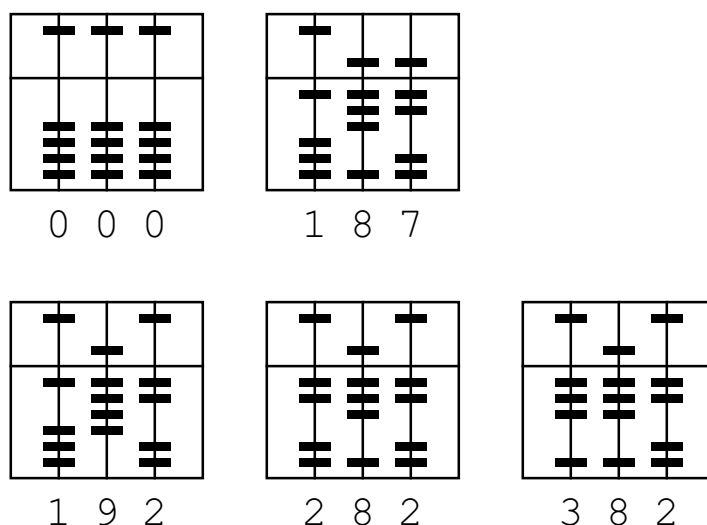


FIGURE 14. Adding $187+195$ on a 1,4 modern soroban.

modified the abacus to the 1,4 modern soroban. This abacus has a unique way of representing each number but requires more mental acuity to use.

The Russian abacus was probably developed in the 17 century and is used so that the rods run horizontally. It contains 10 beads on each rod with one exception which has four beads. To help manage counting the beads rapidly the middle two beads are darker. It is sometime said that there is a bead for each finger with the darker ones representing the thumbs. The schoty was principally used by merchants where the beads on the rod with only four beads represented quarter rubles, the beads above represented one's, ten's, etc. rubles, and those below represented kopeks.

Although there are ways to subtract, to multiply, to divide, to take square and cube roots, and other complicated calculations; we only show the students how to add on the abacus. Let us consider how the sum $187+195$ is found on a modern soroban. First 187 is entered on the abacus, Figure 14. Then we attempt to add the 5 on the one's place but there are not enough beads so we add 1 bead to the 10's place and subtract 5 from the one's place to obtain 192, see Figure 14. Now we try to add 9 beads to the 10's place; however, again we do not have enough beads

in the 10's place so we add 1 bead in the 100's place and subtract one from the 10's place to obtain 282, see Figure 14. Finally we add one to the 100's place to obtain 382, see Figure 14.

In teaching the students how to use the soroban, we do not go into details of how to hold one's fingers when moving the beads, which would ultimately permit them to increase their speed. As a child, one of the authors was taught by a Chinese merchant in Brazil how to use an abacus to add. In spite of hours of practice the author was never able to obtain more praise than "Too slow! You need to work harder!" Then later while visiting Russia in the early 1970's and 1980's two of the authors observed many shop keepers using the schoty with great efficiency. The skill using these devices (like the skill using paper and pencil for calculating) is becoming rarer and rarer as electronic calculators become universally available.

The students are asked to identify China, the most populous country in the world, and its capital, Beijing. They also identify Japan, the number two economic power of the world, along with its capital, Tokyo. Korea is also identified with its capital, Seoul. Ancient Babylonia was in the southern part of modern day Iraq. The date 1200 can be related to Marco Polo's trip from Venice to China circa 1270. Marco Polo was one of the first of the Medieval Europeans to visit the Orient.

6. VEDIC MATHEMATICS

Much of what we know about vedic mathematics is based on a book by Swami Bharati Krishna Tirthaji (1884-1960), which was published after his death. The work describes applications to mainly arithmetic of 16 sutras (concisely stated rules) and 13 sub-sutras taken from the *Parisista* an appendix of the *Atharva Veda*. The *Atharva Veda* is one of the four major *Vedas*, which are among the earliest and most sacred writings of Hinduism dating from between 1000BCE and 500BCE. Although the claim of the existence *Parisista* has never been proved many of the techniques published in Bharati Tirthaji's book are of ancient origin, see [17]. Each of the sutras has many different applications. For a nice set of tutorials for using several of the sutras see [15]. In this activity the students look at two sutras in Sanskrit *Ekadhikena Purvena* and *Nikhilam*.

The first sutra we consider, *Ekadhikena Purvena*, can be translated as *By one more than the previous one* and is applied to two problems of calculation. The first problem considered is that of computing a square of a number ending in a 5, for example 45. The sutra in this application is viewed as a concise statement of the rule *take the digits before the 5 and add 1, then take the product of this number and the original digits and append 25*. Thus to square 45, we compute $4 \times (4 + 1)$ to get 20 and then append 25 to get 2025. As a more complicated example, we compute 105^2 by taking $10 \times (10 + 1) = 110$ and appending 25 we get 11025. This sutra can also be applied to convert quickly a fraction of the form $\frac{1}{x9}$ where x is any digit to decimal [10].

The sutra, *Nikhilam*, can be translated as *All from 9 and the last from 10*. This sutra can be used to multiply using a "base". For example, if one wants to compute 96×88 first pick a base of 100 and then compute $(100 - 96) \times (100 - 88)$ to get $4 \times 12 = 48$. Then by putting the digits $88 - 4 = 84$ on the left we get 8448, which is 96×88 . As another example consider 8×7 . Here we take the base to be 10 and compute $(10 - 8) \times (10 - 7) = 2 \times 3 = 6$. Placing $8 - 3 = 5$ (or $7 - 2 = 5$) on the left we get $56 = 8 \times 7$. As a final example consider multiplying 998×692 . We pick

the base 1000 and compute $(1000 - 998) \times (1000 - 692) = 2 \times 308 = 616$ and then place the digits $692 - 2 = 998 - 308 = 690$ in the front to obtain 690,616 which is 998×692 . In general, if computing $p \times q$ we pick a base b , say the power of 10, 10^n , that makes computing in the head easier. Then we compute the *deficiencies* of the two numbers $b - p = p'$ and $b - q = q'$. To find the final n digits of the answer we compute $p' \times q'$. To compute the initial n digits we compute d where $p - q' = q - p' = b - (p' + q') = d$. Simple algebra reveals that

$$db + p'q' = (p + q - b)b + (b - p)(b - q) = bp + bq - b^2 + b^2 - bp - bq + pq = pq.$$

Thus the trick to using this sutra effectively then is to pick the base that simplifies the work. Finally notice that the English meaning of the sutra is only loosely connected to the algorithm and serves more as a mnemonic device for recalling how the deficiencies are calculated.

Finally we show how the sutra *Nikhilam* is used for squaring numbers. This is best shown by example. Consider 9^2 . Again we pick a base that is a power of 10. Since the deficiency of 9 is $10 - 9 = 1$ we get the first digit of the square to be $(9 - 1)$ and the second digit to be $1^2 = 1$. Thus $9^2 = 81$. Let us consider another example, 13^2 . Again we pick 10 as a base and get a deficiency of -3 so the first digits are $13 - (-3) = 16$ and the last digit is $(-3)^2 = 9$. Thus $13^2 = 169$. One final example illustrates this approach. Consider 93^2 . Picking a base of 100 we see that the deficiency of 93 is 7 and the first two digits are $93 - 7 = 86$ and since $7^2 = 49$ we have that $93^2 = 8649$. In general if we want to compute p^2 we pick a base b and find the deficiency of p to be $b - p = p'$. Then p^2 is computed as $(p - p')b + (p')^2$. From simple algebra we see that

$$(p - p')b + (p')^2 = (p - b + p)b + (b - p)^2 = 2bp - b^2 + b^2 - 2bp + p^2 = p^2.$$

As before the trick to applying the sutra effectively is to pick a base that permits all the calculations to be performed in the head.

In this section students are exposed to Sanskrit, an ancient language that has a common root with all Indo-European languages including English and Spanish. Students are shown how to write their names in Sanskrit. Students are asked to color in India, the second most populous country in the world, on the world map and to identify the capital, New Delhi. The dates of 500BCE and 1000BCE are related to the birth of Christ.

7. REVIEW THE SOCIAL STUDIES INVOLVED

A final activity reviews and expands on the geographical and historical facts that have been mentioned in the other activities. A history line with events important to the methods of calculation along with dates the students are probably familiar with is shown. Capitals of the several countries mentioned are reviewed. The relative sizes of the populations and land masses of assorted large countries are discussed. The students use electronic globes that allow them to obtain distances to various capitals from their hometown. Finally they are each given a small paper globe that they can assemble and take home.

Teachers are given the link to the website developed at the School of Mathematics and Statistics at University of St. Andrews in Scotland [19]. This website contains much historical information along with much biographical information about mathematicians along with maps showing the location of their birth places. Teachers

are also given the link to a website that contains a wonderful timeline that also contains many mathematicians and scientists [18].

8. CONCLUSION

We have described six activities involving arithmetic calculation that can be used with students to increase both their mathematical and cultural awareness. The activities involve enough hands-on work to appeal to most students. The methods of calculation shown raise many questions. Consider, for instance, that the price of a TI-89 calculator today is a little cheaper than the price of a top model K&E slide rule in the 1960's once the price is adjusted for inflation. It is clear that as the price of electronics continues to drop, calculators will eventually completely replace most of the methods calculation described here. It is easy to lament this fact. Skills developed operating slide rules and abacuses will be lost forever. Graduates from high school (and college) will be less and less able to do arithmetic in their heads. This seems to be a hidden cost of technology.

There is a story told about Richard Feynman, the famous physicist, competing with an abacus user. When doing additions it was clear that the abacus was faster. When doing multiplications the abacus was still faster but not by as great a margin. When doing a cube root, however, Feynman was faster because of his deeper knowledge of numbers than the abacus user had. The abacus user was stuck using a demanding algorithm that ignored deeper understanding. The point is that various methods of calculation and technology both demand and allow us to vary the type of thinking in which we engage. The ideal would clearly be to teach our students a large variety of methods of calculation so that they can always use the appropriate method to permit them to think about the most important and illuminating aspect of a problem. In reality it is probably sufficient just to teach them to think about something while they are pushing the buttons on their calculators.

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