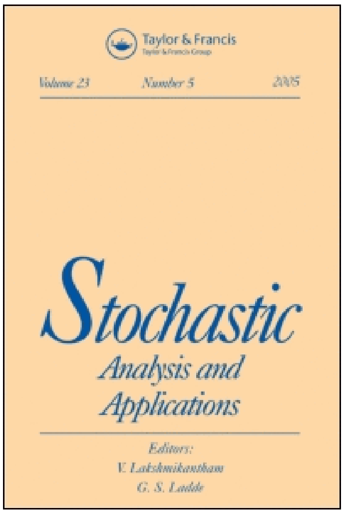

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A Three-Field Finite Element Method for Elliptic Partial Differential Equations Driven by Stochastic Loads

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A Three-Field Finite Element Method for Elliptic Partial Differential Equations Driven by Stochastic Loads

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Abstract: This paper is concerned with the application of nonconforming finite element methods to stochastic partial differential equations. We present a mixed formulation of a three-field finite element method applied to an elliptic model problem involving stochastic loads. We then derive the exact form for the expected value and variance of the solution. Additionally, the rate of convergence for the stochastic error is presented. Finally, we demonstrate through numerical experiments that the method is robust and reliable.

Keywords: Domain decomposition; Finite elements; Nonmatching grids; Stochastic partial differential equation; Three-field.

Mathematics Subject Classification: 35R60; 60H15; 65N30; 65N55.

1. INTRODUCTION

Domain decomposition methods have become an area of significant research and, alongside finite element methods, it has become an essential

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approach to solve coupled physical processes over increasingly complex domains. In broad terms, domain decomposition seeks to analyze complex global domains by decomposing them into several nonoverlapping subdomains and the physical processes are then studied independently over each subdomain. Meshes on these separate components may be available from previous analyses or may be constructed separately by different analysts. During this process, each analyst may employ different finite element discretization procedures and hence the meshes may be differently constructed over each subdomain. This causes the meshes of the subdomains to not conform at the common interface. Once the local solutions have been obtained over each subdomain, they are then assembled to produce a solution over the global domain. Moreover, to obtain faster solutions, the problem may be solved locally over each subdomain. For a very simple example, consider the tensor product domains illustrated in Figure 1.

In other applications, domain decomposition is motivated by discontinuous parameters within the problem. For example, one subgroup in a population model may be governed by one set of environmental factors and a separate nonoverlapping subgroup may be governed by another. In the event that the subgroups are spatially connected, this motivates an individual analysis of each subgroup, but in modeling the entire population, one will "stitch" the subgroups together.

In [10], the authors considered the suitability of three-field methods in three-dimensions for hp implementation, to couple solutions over subdomains with nonmatching grids. We tested the performance of the technique by performing the h-version and the p-version. Specifically, we used tensor product B-splines as the basis functions for our finite element spaces. Other references for the three-field method include [3, 5, 14].

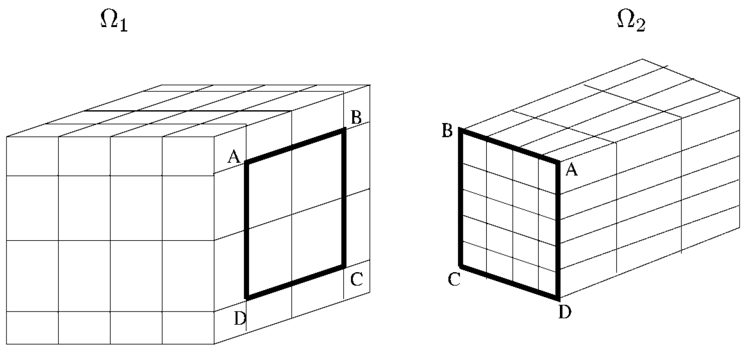

Figure 1. Independently modeled subdomains, Ω_1 and Ω_2, with nonmatching grids on the interface ABCD.

It has also become increasingly acceptable to utilize stochastic differential equation (SDE) models as a reliable component in the analysis of various complex phenomena, ranging from multiphysics (i.e., coupled physical process) to finance. A wide variety of numerical methods and approximation techniques have been developed for SDEs [1, 2, 7, 11, 12, 15, 18]. The motivation of this paper is in the application of domain decomposition methods to stochastic *partial* differential equation (SPDE) models. In this paper, we first present the mixed formulation for the three-field domain decomposition method in [10] and then apply it to an elliptic PDE driven by white noise. In order to implement the method, we use a piecewise constant approximation to finite dimensional white noise and verify an error bound for its integral. We report on the computational issues associated with the stochastic finite element problem, namely, the convergence rate of our stochastic finite element solution. We also derive the exact form for the expected value and variance of the stochastic solution. Finally, we demonstrate the reliability of such a method by presenting various numerical examples for two scenarios, namely, continuous and discontinuous coefficients in the model problem.

2. OVERVIEW OF THE DETERMINISTIC MODEL PROBLEM

We consider the following second-order elliptic problem,

$$\begin{aligned} -\mathrm{div}(a\nabla u) + bu &= f \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega_D, \\ a\frac{\partial u}{\partial n} &= g \quad \text{on } \partial\Omega_N \end{aligned} \tag{1}$$

where $\bar{a}_i \geq a \geq \underline{a}_i > 0$, that is, a is a bounded, uniformly positive function over Ω. In the boundary condition, $a\frac{\partial u}{\partial n}$ is the flux of the solution, u, across the boundary $\partial\Omega_N$. We let $b \geq 0$ in the bounded domain $\Omega \in \mathbb{R}^3$ with boundary $\partial\Omega = \overline{\partial\Omega_D} \cup \overline{\partial\Omega_N}(\partial\Omega_D \cap \partial\Omega_N = \emptyset, \partial\Omega_D \neq \emptyset))$. For $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$, there exists a unique solution

$$u \in H_D^1(\Omega) \equiv \{v \in H^1(\Omega) \,|\, v = 0 \text{ on } \partial\Omega_D\},$$

where we are using $H^k(\Omega)$ to denote the space of functions with k generalized derivatives on Ω. We denote the norm of $H^k(\Omega)$ by $\|\cdot\|_{k,\Omega}$. The definition of these spaces can be extended to noninteger values of k by interpolation. Also note that, for any functional space $X(\Omega)$, the bold symbol $\mathbf{X}(\Omega)$ stands for the product $X(\Omega) \times X(\Omega)$ so that, for instance,

$$\mathbf{H}^k(\Omega) = H^k(\Omega) \times H^k(\Omega).$$

Now, let us assume Ω is the union of S nonoverlapping polygonal subdomains $\{\Omega_i\}_{i=1}^{S}$, such that $\partial\Omega_i \cap \partial\Omega_j (i < j)$ is either empty, a vertex, an edge, or an entire face of Ω_i and Ω_j. Let us for simplicity, assume that the union of all intersections $\partial\Omega_i \cap \partial\Omega_j$, $i < j$ is the face Γ_{ij}. We set the *interface set* Γ to be the union of all Γ_{ij}. We further assume that each Ω_i is a tensor product domain divided into subboxes, i.e., the Cartesian product of three subintervals. It should be noted that the meshes over different Ω_i are independent of each other, with no compatibility enforced across interfaces. Since the meshes are not assumed to conform across interfaces, two separate trace meshes can be defined on Γ_{ij}, one from Ω_i and the other from Ω_j.

Our problem of finding the continuous solution u satisfying (1) becomes solving the following system for u_i, that is, the interior solution variable in each Ω_i,

$$\begin{aligned} -\mathrm{div}(a\nabla u_i) + bu_i &= f_i \quad \text{on } \Omega_i \\ u_i &= 0 \quad \text{on } \partial\Omega_D \cap \partial\Omega_i \\ a\frac{\partial u_i}{\partial n^i} &= g_i \quad \text{on } \partial\Omega_N \cap \partial\Omega_i. \end{aligned} \tag{2}$$

Multiplying the partial differential equation in (2) by a test function $v_i \in H_D^1(\Omega_i)$, integrating by parts, and applying the boundary conditions gives,

$$a_S(u, v) + b_S(v, \lambda) = F(v), \tag{3}$$

where $a_S(u, v)$ is the bilinear form defined by

$$a_S(u, v) = \sum_{i=1}^{S} \int_{\Omega_i} a\nabla u_i.\nabla v_i + bu_i v_i \, dx \tag{4}$$

and $b_S(v, \lambda)$ is the bilinear form defined by

$$b_S(v, \lambda) = \sum_{\Gamma_{ij}\subset\Gamma} \int_{\Gamma_{ij}} \bar{v}_i\lambda^i + \bar{v}_j\lambda^j \, ds. \tag{5}$$

Here $\bar{v}_i$ and $\bar{v}_j$ denote the traces of the function v_i and v_j on Γ_{ij}, respectively. The fluxes at the interface are $\lambda^i = -a\frac{\partial u_i}{\partial n^i} \in H^{-\frac{1}{2}}(\Gamma_{ij})$ and $\lambda^j = -a\frac{\partial u_j}{\partial n^j} \in H^{-\frac{1}{2}}(\Gamma_{ij})$, with n^i and n^j being the corresponding unit outward normals from Ω_i and Ω_j, respectively, at the interface Γ_{ij}. We have that $\lambda \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_{ij}) = H^{-\frac{1}{2}}(\Gamma_{ij}) \times H^{-\frac{1}{2}}(\Gamma_{ij})$. Note that $H^{-\frac{1}{2}}(\Gamma_{ij})$ is the topological dual space of $H^{\frac{1}{2}}(\Gamma_{ij})$.

It must also be noted that (3) must be solved along with the continuity condition enforced on $\bar{u}_i$, the trace of the solution u_i on Γ_{ij} given by

$$\int_{\Gamma_{ij}} \bar{u}_i\chi^i \, ds = \int_{\Gamma_{ij}} t\chi^i \, ds \tag{6}$$

where χ^i is any function in $H^{-\frac{1}{2}}(\Gamma_{ij})$ and t is a new unknown we introduce, called the *interface displacement*, which belongs to $H^{\frac{1}{2}}(\Gamma_{ij})$.

Using (5) and (6), we have:

$$b_S(u, \chi) = \sum_{\Gamma_{ij} \subset \Gamma} \int_{\Gamma_{ij}} (\chi^i + \chi^j) t \, ds. \tag{7}$$

Also, since the solution is smooth in the interior in Ω, we have

$$\lambda^i + \lambda^j = 0,$$

which can be rewritten as,

$$c_S(\lambda, \mu) = 0 \quad \forall \mu \in \prod_{\Gamma_{ij} \subset \Gamma} H^{\frac{1}{2}}(\Gamma_{ij}), \tag{8}$$

where $c_S(\lambda, \mu)$ is a bilinear form defined by

$$c_S(\lambda, \mu) = -\sum_{\Gamma_{ij} \subset \Gamma} \int_{\Gamma_{ij}} (\lambda^i + \lambda^j) \mu \, ds.$$

If we define the space

$$X = \prod_{i=1}^{S} H_D^1(\Omega_i) \times \prod_{\Gamma_{ij} \subset \Gamma} \mathbf{H}^{-\frac{1}{2}}(\Gamma_{ij}) \times \prod_{\Gamma_{ij} \subset \Gamma} H^{\frac{1}{2}}(\Gamma_{ij}),$$

then the problem (1) can now be stated in *mixed form*: Find $(u, \lambda, t) \in X$, such that,

$$\begin{aligned} a_S(u, v) + b_S(v, \lambda) &= F(v) \\ b_S(u, \chi) + c_S(\chi, t) &= 0 \\ c_S(\lambda, \mu) &= 0 \end{aligned} \tag{9}$$

for all $(v, \chi, \mu) \in X$. This problem is given the name the three-field finite element method due to this structure of X.

It has been shown in [5] that for this choice of X, we have a unique solution to (9) for every $f \in L^2(\Omega)$. We also assume that $g \in L^2(\partial\Omega)$ so that the uniqueness follows similarly for the mixed boundary conditions. Moreover, if we have that w is the solution to problem (1) and (u, λ, t) is the solution to (9) then

$$\begin{aligned} u_i &= w \quad \text{on } \Omega_i \\ \lambda^i &= a\frac{\partial w}{\partial n^i} \quad \text{on } \Gamma_{ij} \subset \Gamma \\ t &= w \quad \text{on } \Gamma \end{aligned}$$

where $a\frac{\partial w}{\partial n^i}$ represents the outward normal derivative of the restriction of w to Ω_i, i.e., the flux of w over Γ_{ij}.

Next, we discretize the problem in the mixed formulation (9) by the finite element method. Let $V = \prod_{i=1}^{S} V^i$ such that for each i, $V^i \subset H_D^1(\Omega_i)$ is finite dimensional. Let $\Lambda = \prod_{\Gamma_{ij}\subset\Gamma} \Lambda_i^{\Gamma_{ij}} \times \Lambda_j^{\Gamma_{ij}}$, where $\Lambda_i^{\Gamma_{ij}} \subset H^{-\frac{1}{2}}(\Gamma_{ij})$ and $\Lambda_j^{\Gamma_{ij}} \subset H^{-\frac{1}{2}}(\Gamma_{ij})$ are finite dimensional. Also, let $T = \prod_{\Gamma_{ij}\subset\Gamma} T^{\Gamma_{ij}}$, where $T^{\Gamma_{ij}} \subset H^{\frac{1}{2}}(\Gamma_{ij})$ is finite dimensional. We define $X_N = V \times \Lambda \times T$ so that X_N is a finite dimensional subspace of X.

Our discrete problem can then be stated as follows: Find $(u_N, \lambda_N, t_N) \in X_N$ such that,

$$\begin{aligned} a_S(u_N, v_N) + b_S(v_N, \lambda_N) &= F(v_N) \\ b_S(u_N, \chi_N) + c_S(\chi_N, t_N) &= 0 \\ c_S(\lambda_N, \mu_N) &= 0 \end{aligned} \tag{10}$$

for all $(v_N, \chi_N, \mu_N) \in X$.

It is worth noting that although it is common to use piecewise polynomials as basis elements in such finite element formulations, we have chosen to use tensor product B-splines due to their simplicity in formulation and efficiency of evaluation. Additionally, they form a basis for piecewise polynomials over tensor product domains with certain controls over continuity at element boundaries [8].

As the next natural step in the finite element procedure, we express the unknowns u_N, λ_N, t_N as a linear combination of the respective basis functions. Then, choosing the test functions v_N, χ_N, μ_N to be basis functions themselves then converts the system of integral equations into a matrix system that is solved for the unknowns u_N, λ_N, t_N.

In general, the system will take the form $P\mathbf{c} = \mathbf{F}$, where the solution $\mathbf{c}$ is a vector containing all the coefficients for linear combinations of the unknowns u_N, λ_N, t_N with

$$P = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} \\ \mathbf{B}^T & \mathbf{0} & \mathbf{C} \\ \mathbf{0} & \mathbf{C}^T & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \mathbf{F} = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}.$$

In this form,

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{A}_S \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{B}_S \end{bmatrix},$$

where for each $k = 1, \ldots, S$, $\mathbf{A}_k$ is the square matrix whose (i, j) entry is given by

$$\mathbf{A}_k(i, j) = \int_{\Omega_i} a\nabla v_i \cdot \nabla v_j + b v_i v_j \, dx$$

in which v_i, v_j represent the basis functions for V^k. Thus, the size of each A_k is $\dim(V^k) \times \dim(V^k)$. Also, $\mathbf{B}$ is a block rectangular matrix, where for each $k = 1, \ldots, S$, $\mathbf{B}_k$ has the block form

$$\mathbf{B}_k = \left[\mathbf{B}_{k,\Gamma_{l_1 k}} \cdots \mathbf{B}_{k,\Gamma_{l_L k}}\right]$$

in which L represents the number of interfaces, Γ_{lk}, such that $\Gamma_{lk} \subset \Gamma$ for fixed k. In other words, there is a matrix $\mathbf{B}_{k,\Gamma_{lk}}$ for each interface contained in the boundary of Ω_k. Then, for $\Gamma_{lk} \subset \Gamma$, the (i, j) element of $\mathbf{B}_{k,\Gamma_{lk}}$ is given by

$$\mathbf{B}_{k,\Gamma_{lk}}(i, j) = \int_{\Gamma_{lk}} \lambda_i v_j \, ds,$$

where λ_i are basis functions for $\Lambda_l^{\Gamma_{lk}}$ and v_j are the basis functions for V^k. Thus, matrix $\mathbf{B}_k$ will have size

$$\dim(V^k) \times \left(\sum_{\Gamma_{lk} \subset \Gamma} \dim(\Lambda_k^{\Gamma_{lk}}) \right).$$

Finally, $\mathbf{C}$ has a more complicated form, namely, it is a block matrix with a block column for each distinct $\Gamma_{ij} \subset \Gamma$ and a block row corresponding to each block matrix, $\mathbf{B}_{k,\Gamma_{lk}}$. The matrix $\mathbf{C}_{k,\Gamma_{lk}}$ has an (i, j) element of

$$\mathbf{C}_{k,\Gamma_{lk}}(i, j) = \int_{\Gamma_{lk}} \lambda_i t_j \, ds,$$

where λ_i and t_j are the basis functions for $\Lambda_l^{\Gamma_{lk}}$ and $T^{\Gamma_{lk}}$, respectively. For a given row corresponding to $\mathbf{B}_{k,\Gamma_{lk}}$, $\mathbf{C}_{k,\Gamma_{lk}}$ is placed in the block column corresponding to Γ_{lk} and the remaining block elements of that row are $\mathbf{0}$. Thus, the size of the matrix $\mathbf{C}$ is given by

$$\left(\sum_{k=1}^{S} \sum_{\Gamma_{lk} \subset \Gamma} \dim(\Lambda_k^{\Gamma_{lk}}) \right) \times \left(\sum_{\Gamma_{ij}} \dim(T^{\Gamma_{ij}}) \right).$$

Overall, the total size of the matrix P is $N \times N$, where

$$N = \sum_{k=1}^{S} \dim(V^k) + \sum_{k=1}^{S} \sum_{\Gamma_{lk} \subset \Gamma} \dim(\Lambda_k^{\Gamma_{lk}}) + \sum_{\Gamma_{ij} \subset \Gamma} \dim(T^{\Gamma_{ij}}).$$

Last of all, we note that vector $\mathbf{F}$ has the form

$$\mathbf{F}_1 = \begin{bmatrix} \mathbf{F}_{1,1} \\ \mathbf{F}_{1,2} \\ \vdots \\ \mathbf{F}_{1,S} \end{bmatrix},$$

such that $\mathbf{F}_{1,k}$ is a vector of length $\dim(V^k)$ with its ith entry given by

$$\mathbf{F}_{1,k}(i) = \int_{\Omega_k} f v_i \, dx + \int_{\partial\Omega_k \cap \partial\Omega_N} g v_i \, ds,$$

where v_i are the basis functions for V^k.

In [14], the invertibility of the matrix, P, was considered and it was shown under reasonable assumptions, a unique solution exists to this linear system.

The solution $\mathbf{u}_N = (u_N, \lambda_N, t_N) \in X_N$ is made up of linear combinations of basis functions from each of the spaces,

$$\begin{gathered} V^i, \quad i = 1, \dots, S \\ \Lambda_i^{\Gamma_{ij}} \quad \text{and} \quad \Lambda_j^{\Gamma_{ij}} \quad \forall \Gamma_{ij} \subset \Gamma \\ T^{\Gamma_{ij}} \quad \forall \Gamma_{ij} \subset \Gamma. \end{gathered}$$

Let $\{\phi_i\}_{i=1}^N$ be the set of all basis functions from these spaces, so that we can write

$$\mathbf{u}_N = \sum_{i=1}^N c_i \phi_i,$$

where $\mathbf{c} = P^{-1}\mathbf{F}$.

3. ELLIPTIC PDEs WITH STOCHASTIC LOAD

We consider now the same classical problem as before with one slight modification. Let $(Y, \mathcal{F}, \mathbf{P})$ be a complete probability space. We want to find a stochastic function $u : Y \times \Omega \to \mathbb{R}$ such that, almost surely (a.s.), we have

$$\begin{aligned} -\mathrm{div}(a\nabla u) + bu &= f + r\dot{W} \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega_D \\ a\frac{\partial u}{\partial n} &= g \quad \text{on } \partial\Omega_N, \end{aligned} \tag{11}$$

where W is a Brownian sheet defined on Ω. As before, we assume a is a bounded, uniformly positive function and $b \geq 0$ in the bounded domain, $\Omega \subset \mathbb{R}^3$. Additionally, we assume $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega_N)$ in order to guarantee uniqueness and existence to the deterministic problem. Finally, let $r \in L^2(\Omega)$. The last term may result, in for example, population modeling, where $u(x, y, z)$ is the population density at the point (x, y, z) (see, e.g., [13, 17]). For all derivations in this section, we make the assumption that Ω is a tensor product domain. Additionally, for convenience, we assume $\partial\Omega = \partial\Omega_D$, although this is not necessary and all the results hold otherwise.

The solutions to a general stochastic elliptic problem are discussed in [2]. We define the general tensor product Hilbert space, $\widetilde{H}(\Omega)$, by

$$\begin{aligned}\widetilde{H}(\Omega) &= L^2(Y; H(\Omega))\\ &= \{v \in Y \times \Omega \to \mathbb{R} \mid v \text{ is strongly measurable and}\\ &\qquad E\big[\|v(\cdot, \mathbf{x})\|_H^2\big] < \infty\}.\end{aligned}$$

The space $L^2(Y; H(\Omega))$ is called a tensor product Hilbert space as it is can be shown to be isomorphic to the tensor product Hilbert space, $L^2(Y) \otimes H(\Omega)$. We consider the space $\widetilde{H}^1_D(\Omega)$ endowed with the inner product, $(v, u)_{\widetilde{H}} \equiv E\big[\int_\Omega \nabla v \cdot \nabla u\, dx\big]$. It can be shown by the standard application of the Lax-Milgram Lemma, cf. [4], that there exists a unique solution, $u \in \widetilde{H}^1_D(\Omega)$, such that

$$\mathscr{B}(u, v) = L(v) \quad \forall\, v \in \widetilde{H}^1_D(\Omega) \tag{12}$$

where

$$\begin{aligned}\mathscr{B}(u, v) &= E\Big[\int_\Omega a\nabla u \cdot \nabla u + buv\, dx\Big] \quad \text{and}\\ L(v) &= E\Big[\int_\Omega fv\, dx + \int_\Omega rv\, dW\Big].\end{aligned}$$

By these assumptions on a, f, and r, we have that $\mathscr{B}$ is continuous and coercive and L is a bounded linear functional. Then, the standard arguments from measure theory show that the solution to (12) also solves (11).

In the formulation of a three-field mixed method, we arrive at similar derivations as we have seen in previous sections. The problem (11) can be stated in the following *mixed form*: Find $(u, \lambda, t) \in \prod_{i=1}^S \widetilde{H}^1_D(\Omega_i) \times \prod_{\Gamma_{ij}\subset\Gamma} \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma_{ij}) \times \prod_{\Gamma_{ij}\subset\Gamma} \widetilde{H}^{\frac{1}{2}}(\Gamma_{ij})$ such that,

$$\begin{aligned}a_S(u, v) + b_S(v, \lambda) &= F(v) + N(v)\\ b_S(u, \chi) + c_S(\chi, t) &= 0\\ c_S(\lambda, \mu) &= 0\end{aligned} \tag{13}$$

for all $(v, \chi, \mu) \in \prod_{i=1}^S \widetilde{H}^1_D(\Omega_i) \times \prod_{\Gamma_{ij}\subset\Gamma} \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma_{ij}) \times \prod_{\Gamma_{ij}\subset\Gamma} \widetilde{H}^{\frac{1}{2}}(\Gamma_{ij})$. Notice that in comparison to the mixed form (9), the only difference is the right-hand side of the first equation now contains a stochastic term, $N(v) = \int_\Omega r(\mathbf{x})v(\mathbf{x})dW(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^3$.

We show there exists a unique solution to this mixed formulation following the presentation of Lemma 3.2, using approximations of the Brownian motion. Now, we discretize the problem by letting $u_N \in V \subset \prod_{i=1}^{S} \widetilde{H}_D^1(\Omega_i)$, $\lambda_N \in \Lambda \subset \prod_{\Gamma_{ij}\subset\Gamma} \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma_{ij})$, $t_N \in T \subset \prod_{\Gamma_{ij}\subset\Gamma} \widetilde{H}^{\frac{1}{2}}(\Gamma_{ij})$. Our discrete problem may now be stated as follows: Find $(u_N, \lambda_N, t_N) \in V \times \Lambda \times T$ such that,

$$a_S(u_N, v_N) + b_S(v_N, \lambda_N) = F(v_N) + N(v_N) \tag{14}$$

$$b_S(u_N, \chi_N) + c_S(\chi_N, t_N) = 0 \tag{15}$$

$$c_S(\lambda_N, \mu_N) = 0 \tag{16}$$

for all $(v_N, \chi_N, \mu_N) \in V \times \Lambda \times T$. Again, the only modification from the last section is the addition of a stochastic term to the right-hand side of (14). Notice that upon implementation the stiffness matrix is the same for the stochastic problem as for the deterministic version. This will allow us to calculate a single stiffness matrix and perform a factorization that can be used to efficiently solve the stochastic problem a large number of times. Additionally, the invertibility of the matrix as presented in the previous section yields a unique solution to this linear system. In fact, the linear system now has the form

$$P\mathbf{c} = \mathbf{F} + \mathbf{N},$$

where

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad \text{and} \quad \mathbf{N}_1 = \begin{bmatrix} \mathbf{N}_{1,1} \\ \mathbf{N}_{1,2} \\ \vdots \\ \mathbf{N}_{1,S} \end{bmatrix},$$

such that for each $k = 1, \dots, S$, we have that $\mathbf{N}_{1,k}$ is a vector of length $\dim(V^k)$ whose ith entry is given by

$$\mathbf{N}_{1,k}(i) = \int_\Omega r(\mathbf{x}) v_i(\mathbf{x}) dW(\mathbf{x}).$$

In this case, the functions v_i are the basis functions from V^k. This implies that our solution will have the form

$$\tilde{\mathbf{u}}_N = \sum_{i=1}^{N} \tilde{c}_i \phi_i$$

where $\tilde{\mathbf{c}} = P^{-1}(\mathbf{F} + \mathbf{N})$. Here we note that $\tilde{\mathbf{c}}$ is a random variable since it depends on the stochastic integral in $\mathbf{N}$.

At this point, we will use an approximation of the white noise process by a piecewise constant random function, allowing ease of implementation as well as required smoothness for convergence of numerical methods. Let us consider the finite dimensional white noise, $\dot{W}(\mathbf{x})$. We consider the tensor product domain $\Omega = \prod_{i=1}^{d}[a_i, b_i] \subset \mathbb{R}^d$ and partition each subinterval $[a_i, b_i]$ by

$$a_i = x_{i,1} < x_{i,2} < \cdots < x_{i,n_i} = b_i.$$

For $\mathbf{k} = (k_1, \ldots, k_d)$, such that $1 \leq k_i < n_i$ for each $i = 1, \ldots, d$, we let

$$R_{\mathbf{k}} = \prod_{i=1}^{d}[x_{i,k_i}, x_{i,k_i+1}].$$

Thus, $\{R_{\mathbf{k}}\}$ forms a partition of Ω. We denote the volume of $R_{\mathbf{k}}$ by $|R_{\mathbf{k}}|$. We approximate the white noise by the following piecewise constant random function [1]

$$\hat{\dot{W}}(\mathbf{x}) = \frac{1}{|R_{\mathbf{k}}|}\sum_{\mathbf{k}} \eta_{\mathbf{k}}\sqrt{|R_{\mathbf{k}}|}\chi_{\mathbf{k}}(\mathbf{x}) \tag{17}$$

where

$$\eta_{\mathbf{k}}\sqrt{|R_{\mathbf{k}}|} = \int_{R_{\mathbf{k}}} dW(\mathbf{x}). \tag{18}$$

This sum is taken over all $\mathbf{k} = (k_1, \ldots, k_d)$, such that $1 \leq k_i < n_i$ for all $i = 1, \ldots, d$. In other words,

$$\eta_{\mathbf{k}} \sim N(0, 1) \tag{19}$$

and

$$\chi_{\mathbf{k}}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in R_{\mathbf{k}} \\ 0 & \text{otherwise.} \end{cases}$$

This allows a convenient calculation of an approximation for $\int_{\Omega} f(\mathbf{x})dW(\mathbf{x})$, where $f(\mathbf{x}) \in L^2(\Omega)$. Namely, for a given partition $\{R_{\mathbf{k}}\}$ of Ω, we have

$$\int_{\Omega} f(\mathbf{x})dW(\mathbf{x}) \approx \int_{\Omega} f(\mathbf{x})d\widehat{W}(\mathbf{x}) = \sum_{\mathbf{k}} \frac{\eta_{\mathbf{k}}}{\sqrt{|R_{\mathbf{x}}|}}\int_{R_{\mathbf{k}}} f(\mathbf{x})d\mathbf{x}.$$

In fact, we can prove the result in Lemma 3.2. First, we state some useful properties of the stochastic integral under consideration. These are widely used and follow from the standard argument using sequences of simple functions [1, 19].

Lemma 3.1. *Let $f, g \in L^2(\Omega)$ be nonrandom functions. Then*

1. $E\left[\int_\Omega f(\mathbf{x})dW(\mathbf{x})\right] = 0.$
2. $E\left[\int_\Omega f(\mathbf{x})dW(\mathbf{x})\int_\Omega g(\mathbf{x})dW(\mathbf{x})\right] = \int_\Omega f(\mathbf{x})g(\mathbf{x})d\mathbf{x}.$
3. *Consequently, if we let $f = g$ then we have,*

$$E\left[\int_\Omega f(\mathbf{x})dW(\mathbf{x})\right]^2 = \int_\Omega (f(\mathbf{x}))^2 d\mathbf{x}.$$

We can now prove the following bound on mean square error in our stochastic integral approximation.

Lemma 3.2. *Let $f \in C^1(\Omega)$, where $\Omega = \prod_{i=1}^d [a_i, b_i] \subset \mathbb{R}^d$. Then*

$$E\left[\int_\Omega f(\mathbf{x})dW(\mathbf{x}) - \int_\Omega f(\mathbf{x})d\widehat{W}(\mathbf{x})\right]^2 \leq C\left(\sum_{i=1}^d (\Delta x_i)^2\right),$$

where $C = \frac{7d}{60}|\Omega| \max_{1\leq i\leq d} \|\partial_i f\|_\infty^2$, Δx_i is the subinterval length for each i and $|\Omega|$ denotes the volume of Ω.

Proof. Let $\{R_\mathbf{k}\}$ be a partition of the tensor product domain Ω. For simplicity, let us use uniform partitions with interval lengths Δx_i for each spatial dimension, $i = 1 \ldots d$. Thus, for each $\mathbf{k} = (k_1, \ldots, k_d)$, such that $1 \leq k_i < n_i$, the volume of $R_\mathbf{k}$ is given by $|R_\mathbf{k}| = \prod_{i=1}^d \Delta x_i$. We now let $\eta_\mathbf{k} \sim N(0, 1)$ for each $\mathbf{k}$.

Having established notation, we can see that,

$$\begin{aligned} &E\left[\int_\Omega f(\mathbf{x})dW(\mathbf{x}) - \int_\Omega f(\mathbf{x})d\widehat{W}(\mathbf{x})\right]^2 \\ &\quad = E\left[\int_\Omega f(\mathbf{x})dW(\mathbf{x}) - \sum_\mathbf{k} \frac{\eta_\mathbf{k}}{\sqrt{|R_\mathbf{k}|}}\int_{R_\mathbf{k}} f(\mathbf{x})d\mathbf{x}\right]^2. \end{aligned} \tag{20}$$

Let $\bar{f}_\mathbf{k}$ be the average value of f over $R_\mathbf{k}$, namely,

$$\bar{f}_\mathbf{k} = \frac{1}{|R_\mathbf{k}|}\int_{R_\mathbf{k}} f(\mathbf{x})d\mathbf{x}.$$

The right-hand side of (20) can be rewritten as

$$E\left[\int_\Omega f(\mathbf{x})dW(\mathbf{x}) - \sum_\mathbf{k} \eta_\mathbf{k}\sqrt{|R_\mathbf{k}|}\bar{f}_\mathbf{k}\right]^2.$$

Now by (18), we have

$$\begin{aligned} E\left[\int_\Omega f(\mathbf{x})dW(\mathbf{x}) - \int_\Omega f(\mathbf{x})d\widehat{W}(\mathbf{x})\right]^2 &= E\left[\sum_\mathbf{k}\int_{R_\mathbf{k}} (f(\mathbf{x}) - \bar{f}_\mathbf{k})dW(\mathbf{x})\right]^2 \\ &= \sum_\mathbf{k}\int_{R_\mathbf{k}} (f(\mathbf{x}) - \bar{f}_\mathbf{k})^2 d\mathbf{x}. \end{aligned}$$

Notice that we can say the following,

$$f(\mathbf{x}) - \bar{f}_{\mathbf{k}} = \frac{1}{|R_{\mathbf{k}}|}\int_{R_{\mathbf{k}}}(f(\mathbf{x}) - f(\mathbf{y}))d\mathbf{y}. \tag{21}$$

By application of Taylor's Theorem in the multivariate case, we know that for $f \in C^1(\Omega)$, there exists some $t_{\mathbf{x}} \in (0, 1)$, where if $\xi_{\mathbf{x}} = t_{\mathbf{x}}\mathbf{x} + (1 - t_{\mathbf{x}})\mathbf{y}$, then

$$f(\mathbf{x}) = f(\mathbf{y}) + \sum_{i=1}^{d} \partial_i f(\xi_{\mathbf{x}})(x_i - y_i)^{\alpha}. \tag{22}$$

Thus, by (21) and (22),

$$f(\mathbf{x}) - \bar{f}_{\mathbf{k}} = \frac{1}{|R_{\mathbf{k}}|}\int_{R_{\mathbf{k}}}\sum_{i=1}^{d}\partial_i f(\xi_{\mathbf{x}})(x_i - y_i)d\mathbf{y},$$

which yields

$$\sum_{\mathbf{k}}\int_{R_{\mathbf{k}}}(f(\mathbf{x}) - \bar{f}_{\mathbf{k}})^2 d\mathbf{x} \le \sum_{\mathbf{k}}\int_{R_{\mathbf{k}}}\frac{1}{|R_{\mathbf{k}}|^2}\left(\int_{R_{\mathbf{k}}}\sum_{i=1}^{d}|\partial_i f(\xi_{\mathbf{x}})|\ |x_i - y_i|\ d\mathbf{y}\right)^2 d\mathbf{x}.$$

It is easily shown that for any set of values $\{b_i\}_{i=1}^{d}$,

$$\left(\sum_{i=1}^{d} b_i\right)^2 \le d\sum_{i=1}^{d} b_i^2.$$

By applying this result and taking the sup-norm of $\partial_i f(\xi_{\mathbf{x}})$, we have

$$\sum_{\mathbf{k}}\int_{R_{\mathbf{k}}}(f(\mathbf{x}) - \bar{f}_{\mathbf{k}})^2 d\mathbf{x} \le \sum_{\mathbf{k}}\frac{d}{|R_{\mathbf{k}}|^2}\sum_{i=1}^{d}\|\partial_i f\|_{\infty}^2\int_{R_{\mathbf{k}}}\left(\int_{R_{\mathbf{k}}}|x_i - y_i|\ d\mathbf{y}\right)^2 d\mathbf{x}. \tag{23}$$

We now integrate this directly by expanding the inner integral, namely, for each $i = 1, \ldots, d$ and each $\mathbf{k}$,

$$\begin{aligned}
&\int_{R_{\mathbf{k}}}\left(\int_{R_{\mathbf{k}}}|x_i - y_i|\ d\mathbf{y}\right)^2 d\mathbf{x} \\
&\quad = \prod_{\substack{j=1 \\ i\ne j}}^{d}(\Delta x_i)^2\int_{R_{\mathbf{k}}}\left(\int_{x_{i,k_i}}^{x_{i,k_i+1}}|x_i - y_i|\ dy_i\right)^2 d\mathbf{x} \\
&\quad = \prod_{\substack{j=1 \\ i\ne j}}^{d}(\Delta x_i)^2\int_{R_{\mathbf{k}}}\left(\int_{x_{i,k_i}}^{x_i}(x_i - y_i)dy_i + \int_{x_i}^{x_{i,k_i+1}}(y_i - x_i)dy_i\right)^2 d\mathbf{x} \\
&\quad = \frac{7}{60}|R_{\mathbf{k}}|^3(\Delta x_i)^2.
\end{aligned}$$

Thus, by applying this to equation (23), we have

$$\sum_{\mathbf{k}} \int_{R_{\mathbf{k}}} (f(\mathbf{x}) - \bar{f}_{\mathbf{k}})^2 d\mathbf{x} \leq \frac{7d}{60} \sum_{\mathbf{k}} \sum_{i=1}^{d} \|\partial_i f\|_\infty^2 (\Delta x_i)^2 |R_{\mathbf{k}}|.$$

Using the fact that $\sum_{\mathbf{k}} |R_{\mathbf{k}}| = |\Omega|$, we have the desired result. □

Corollary 3.1. *Let $f \in C^1(\Omega)$, where $\Omega \subset \mathbb{R}^3$ is a tensor product domain. Then*

$$E\left[\int_\Omega f(\mathbf{x}) dW(\mathbf{x}) - \int_\Omega f(\mathbf{x}) d\widehat{W}(\mathbf{x})\right]^2 \leq \frac{7}{20}|\Omega| \max_{1\leq i\leq d} \|\partial_i f\|_\infty^2 \left(\sum_{i=1}^{d} (\Delta x_i)^2\right).$$

Remark 3.1. Note that this result is sharper than the bound given in [1].

Now, we return to the solution of the problem in (13). To show that a solution exists, we construct a sequence of processes $\{W^n(\mathbf{x}) \mid n \in \mathbb{N}\}$ with trajectories in $L^2(\Omega)$ as in [6]. We construct this sequence by letting $\{R_{ijk}\}$ be a partition of Ω with $\Delta x = \Delta y = \Delta z = \frac{1}{n}$ and consider

$$\dot{W}^n(\mathbf{x}) = \frac{1}{|R_{ijk}|} \sum_{ijk} \eta_{ijk} \sqrt{|R_{ijk}|} \chi_{ijk}(\mathbf{x}).$$

We denote

$$N_n(v) = \int_\Omega r(\mathbf{x}) v(\mathbf{x}) dW^n(\mathbf{x}).$$

Thus, by Lemma 3.2, $E[N_n(v) - N(v)]^2 \leq \frac{C}{n^2}$. It can be shown that a solution to (13) exists by replacing $N(v)$ with $N_n(v)$ and showing that the corresponding solutions $\{u_n\}$ form a Cauchy sequence converging in $\widetilde{H}_D^1(\Omega)$ [9]. Uniqueness follows by assuming that two such solutions exist and subtracting the resulting equations in (13). In other words, for $F(v) = 0$, it follows that $u = 0$, $\lambda = 0$, and $t = 0$, a.s.

Recall that we have denoted the solution to (10) by

$$\mathbf{u}_N = \sum_{i=1}^{N} c_i \phi_i(\mathbf{x}),$$

where $\mathbf{c} = P^{-1}\mathbf{F}$. The solution to (14)–(16) is given by

$$\tilde{\mathbf{u}}_N = \sum_{i=1}^{N} \tilde{c}_i \phi_i(\mathbf{x}),$$

where $\tilde{\mathbf{c}} = P^{-1}(\mathbf{F} + \mathbf{N})$. We now prove the following theorem.

Theorem 3.1. *$E[\tilde{\mathbf{c}}] = \mathbf{c}$ and therefore, $E[\tilde{\mathbf{u}}_N(\mathbf{x})] = \mathbf{u}_N(\mathbf{x})\ \forall\, \mathbf{x} \in \Omega$.*

Proof. Clearly, we have the linear system $P\tilde{\mathbf{c}} = \mathbf{F} + \mathbf{N}$, where P is the stiffness matrix. This implies that

$$\begin{aligned} E[\tilde{\mathbf{c}}] &= E[P^{-1}(\mathbf{F} + \mathbf{N})] \\ &= P^{-1}(\mathbf{F} + E[\mathbf{N}]) \\ &= P^{-1}\mathbf{F} \\ &= \mathbf{c}. \end{aligned}$$

Here, we have used the first property of the stochastic integral stated in Lemma 3.1, yielding $E[\mathbf{N}] = 0$. Now, since $\tilde{\mathbf{u}}_N$ and $\mathbf{u}_N$ are simply linear combinations of the same set of basis functions, then $E[\tilde{\mathbf{u}}_N] = \mathbf{u}_N$. □

One can also show the coefficients of the stochastic solution satisfy the following theorem. This allows us to compare the computed results with a known theoretical result.

Theorem 3.2. *$\mathrm{Var}(\tilde{\mathbf{c}}) = P^{-1}\widehat{H}P^{-1}$ such that $\widehat{H}$ is the block matrix given by*

$$\widehat{H} = \begin{bmatrix} H & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

where H is a block diagonal matrix of size

$$\left(\sum_{k=1}^{S} \dim(V^k)\right) \times \left(\sum_{k=1}^{S} \dim(V^k)\right).$$

For each matrix H_k, $k = 1, \ldots, S$, on the diagonal of H, the (i, j) entry of H_k is given by

$$H_k(i, j) = \int_\Omega (r(\mathbf{x}))^2 v_i(\mathbf{x}) v_j(\mathbf{x}) d\mathbf{x},$$

where v_i and v_j represent the basis functions for V^k.

Proof (Theorem 3.2). Clearly, since $\mathbf{c} = P^{-1}\mathbf{F}$ is a constant vector, then $\mathrm{Var}(\tilde{\mathbf{c}}) = \mathrm{Var}(\tilde{\mathbf{c}} - \mathbf{c})$. Now using the fact that $\tilde{\mathbf{c}} = P^{-1}(\mathbf{F} + \mathbf{N})$, we have that $\mathrm{Var}(\tilde{\mathbf{c}} - \mathbf{c}) = \mathrm{Var}(P^{-1}\mathbf{N})$. By the definition of the covariance matrix of a vector, we see that for any vector $\mathbf{z}$ and any constant matrix B, $\mathrm{Var}(B\mathbf{z}) = B\mathrm{Var}(\mathbf{z})B^T$. Therefore, using the symmetry of P (i.e., $P^{-1} = (P^{-1})^T$), we now have that $\mathrm{Var}(P^{-1}\mathbf{N}) = P^{-1}\mathrm{Var}(\mathbf{N})P^{-1}$.

It then remains to be proved that $\mathrm{Var}(\mathbf{N}) = \widehat{H}$. We use the definition of the covariance matrix and the fact that $E[\mathbf{N}] = 0$ to obtain the following

$$\mathrm{Var}(\mathbf{N}) = E[\mathbf{N}\mathbf{N}^T] = E\begin{bmatrix} H & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

where, by simple vector multiplication, H is a diagonal block matrix of size

$$\left(\sum_{k=1}^{S}\dim(V^k)\right)\times\left(\sum_{k=1}^{S}\dim(V^k)\right).$$

For each matrix H_k, $k = 1, \ldots, S$, on the diagonal of H, the (i, j) entry of H_k is given by

$$H_k(i, j) = \int_\Omega r(\mathbf{x})v_i(\mathbf{x})dW(\mathbf{x})\int_\Omega r(\mathbf{x})v_j(\mathbf{x})dW(\mathbf{x}),$$

where v_i and v_j represent the basis functions for V^k. By the second property of the stochastic integral in Lemma 3.1, we can say that the (i, j) entry of H_k is given by

$$H_k(i, j) = \int_\Omega (r(\mathbf{x}))^2\phi_i(\mathbf{x})\phi_j(\mathbf{x})d\mathbf{x}.$$

Thus, $\mathrm{Var}(\mathbf{N}) = \widehat{H}$. □

This theorem gives us a theoretical result that will be used later to compare any computed numerical results. In addition to examining the variance of the coefficients, it will be useful to consider the variance of the solution at a given fixed point within the domain Ω. Fortunately, the point-wise variance satisfies the following theorem.

Theorem 3.3. *For any fixed point* $\mathbf{x}\in\Omega$,

$$\mathrm{Var}\big(\tilde{\mathbf{u}}_N(\mathbf{x})\big) = \Phi(\mathbf{x})^T\mathrm{Var}(\tilde{\mathbf{c}})\Phi(\mathbf{x}),$$

where $\Phi(\mathbf{x})$ *is the vector whose elements are the point-wise evaluation of each basis function of* X_N *at the point* $\mathbf{x}$.

Proof. Since $E[\tilde{\mathbf{u}}_N(\mathbf{x})] = \mathbf{u}_N(\mathbf{x})$, we have

$$\begin{aligned}
\mathrm{Var}(\tilde{\mathbf{u}}_N(\mathbf{x})) &= E[\tilde{\mathbf{u}}_N(\mathbf{x}) - \mathbf{u}_N(\mathbf{x})]^2\\
&= E\left[\sum_{i=1}^{N}(\tilde{c}_i - c_i)\phi_i(\mathbf{x})\right]^2\\
&= E\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\phi_i(\mathbf{x})(\tilde{c}_i - c_i)(\tilde{c}_j - c_j)\phi_j(\mathbf{x})\right]\\
&= (\Phi(\mathbf{x}))^T E\big[(\tilde{\mathbf{c}}-\mathbf{c})(\tilde{\mathbf{c}}-\mathbf{c})^T\big]\Phi(\mathbf{x})\\
&= \big(\Phi(\mathbf{x})\big)^T\mathrm{Var}(\tilde{\mathbf{c}})\Phi(\mathbf{x}).
\end{aligned}$$

□

Let us now, for convenience, denote a single realization of the stochastic solution by $\tilde{\mathbf{u}}_N^{(j)}$, and likewise, its coefficients are $\{\tilde{c}_i^{(j)}\}$. We now consider a theoretical result on the statistical error, which we will denote by

$$e_S = E(\tilde{\mathbf{u}}_N) - \frac{1}{M}\sum_{j=1}^{M}\tilde{\mathbf{u}}_N^{(j)} = \mathbf{u}_N - \frac{1}{M}\sum_{j=1}^{M}\tilde{\mathbf{u}}_N^{(j)}$$

where M is the number of realizations. Note that for each $j = 1, \ldots M$, $\tilde{\mathbf{u}}_N^{(j)}$ is a unique realization of the solution to the stochastic finite element problem. Each $\tilde{\mathbf{u}}_N^{(j)}$ depends on the generation of the random values for $\eta_{\mathbf{k}}$ in the approximation of stochastic integral on the right-hand side of (14). The following theorem describes the behavior of $\|e_S\|_{1,\Omega}$ as $M \to \infty$. The proof given utilizes the finite dimensionality of our approximation spaces.

Theorem 3.4. *If we let* $e_S = \mathbf{u}_N - \frac{1}{M}\sum_{j=1}^{M}\tilde{\mathbf{u}}_N^{(j)}$, *where* $M \in \{2^k \mid k \in \mathbb{N}\}$, *then for* $\alpha \in (0, 1/2)$ *and any fixed mesh, h,*

$$\lim_{M\to\infty} M^{\alpha}\|e_S\|_{1,\Omega} = 0 \quad a.s.$$

Proof. First, we verify the claim that for each fixed finite element mesh, h, there exists a C independent of M, such that

$$E(\|e_S\|_{1,\Omega}) \le \frac{C}{M} \quad \forall M \in \mathbb{N}.$$

Here we use the fact that $\mathbf{u}_N = \sum_i c_i\phi_i(x)$ and for each realization, $\tilde{\mathbf{u}}_N^{(j)} = \sum_i \tilde{c}_i^{(j)}\phi_i(x)$. Thus,

$$\begin{aligned} e_s &= \mathbf{u}_N - \frac{1}{M}\sum_{j=1}^{M}\tilde{\mathbf{u}}_N^{(j)} \\ &= \sum_{i=1}^{N} c_i\phi_i(x) - \frac{1}{M}\sum_{j=1}^{M}\left(\sum_{i=1}^{N}\tilde{c}_i^{(j)}\phi_i(x)\right) \\ &= \sum_{i=1}^{N}\left(c_i - \frac{1}{M}\sum_{j=1}^{M}\tilde{c}_i^{(j)}\right)\phi_i(x). \end{aligned}$$

Now we denote $\sigma_i = \mathrm{Var}(\tilde{c}_i)$ and note the fact that $E(\tilde{c}_i) = c_i$. Therefore, by the sampling distribution of the means,

$$E\left(c_i - \frac{1}{M}\sum_{j=1}^{M}\tilde{c}_i^{(j)}\right)^2 = \frac{\sigma_i^2}{M}.$$

Now, consider that

$$E(\|e_S\|_{1,\Omega}^2) = E\left(\left\| \sum_{i=1}^{N}\left(c_i - \frac{1}{M}\sum_{j=1}^{M}\tilde{c}_i^{(j)}\right)\phi_i(x)\right\|_{1,\Omega}^2\right).$$

Note that

$$\left\| \sum_{i=1}^{n}\left(c_i - \frac{1}{M}\sum_{j=1}^{M}\tilde{c}_i^{(j)}\right)\phi_i(x)\right\|_{1,\Omega}^2 \leq \sum_{i=1}^{N}\left|c_i - \frac{1}{M}\sum_{j=1}^{M}\tilde{c}_i^{(j)}\right|^2 \|\phi_i\|_{1,\Omega}^2.$$

Thus,

$$\begin{aligned} E(\|e_S\|_{1,\Omega}^2) &\leq E\left(\sum_{i=1}^{N}\left|c_i - \frac{1}{M}\sum_{j=1}^{M}\tilde{c}_i^{(j)}\right|^2 \|\phi_i\|_{1,\Omega}^2\right) \\ &= \sum_{i=1}^{N} E\left(\left|c_i - \frac{1}{M}\sum_{j=1}^{M}\tilde{c}_i^{(j)}\right|^2\right)\|\phi_i\|_{1,\Omega}^2, \\ &= \frac{1}{M}\sum_{i=1}^{N}\sigma_i^2\|\phi_i\|_{1,\Omega}^2 \leq \frac{1}{M}C \end{aligned}$$

where $C < \infty$ and C is independent of M. This follows from properties of B-splines and that $\{\sigma_i \mid i = 1, \ldots, n\}$ is finite.

Now we follow a similar argument as in [2]. Let $\varepsilon > 0$ and by the Markov Inequality

$$\begin{aligned} \mathrm{Prob}\left(\|e_S\|_{1,\Omega} > \frac{\varepsilon}{M^\alpha}\right) &\leq \frac{M^{2\alpha}}{\varepsilon^2}E(\|e_S\|_{1,\Omega}^2) \\ &\leq \frac{C}{\varepsilon^2 M^{1-2\alpha}}. \end{aligned}$$

We choose an increasing sequence of realizations $\{M_k\}_{k=1}^{\infty}$, where $M_k \in \{2^n \mid n \in \mathbb{N}\}$, for each k. Then,

$$\begin{aligned} \sum_{k=1}^{\infty}\mathrm{Prob}\left(\|e_S\|_{1,\Omega} > \frac{\varepsilon}{M^\alpha}\right) &\leq \frac{C}{\varepsilon^2}\sum_{k=1}^{\infty}\frac{1}{M_k^{1-2\alpha}} \\ &\leq \frac{C}{\varepsilon^2}\sum_{k=1}^{\infty}\frac{1}{2^{(1-2\alpha)k}}, \\ &< \infty \end{aligned}$$

where the final inequality holds when $\alpha \in (0, 1/2)$. Thus, by the Borel-Cantelli Lemma [16],

$$\lim_{m\to\infty} M^\alpha\|e_S\|_{1,\Omega} = 0 \quad \text{a.s.} \qquad \square$$

Table 1. *M*-version for continuous a with $(p_1, p_2, q_1, q_2, r) = (4, 4, 3, 3, 3)$, $(m, n) = (4, 6)$ and $M = 10^k, k = 1, \dots, 6$

M	$\left\| E(\tilde{u}_N^{(j)}) - u_N \right\|_{1,\Omega}$
10	0.26405878
100	0.06725450
1000	0.01890008
10000	0.00703104
100000	0.00193773
1000000	0.00085414
Observed Rate	$O(M^{-1/2})$

4. NUMERICAL RESULTS FOR STOCHASTIC DRIVEN ELLIPTIC PDEs

In this section, we demonstrate the result of implementing the model problem (11) over the domain $[0, 2] \times [0, 1] \times [0, 1]$. We partition this domain into two subdomains, Ω_1 and Ω_2, by the interface plane $x = 1$. For simplicity, we implement using only Dirichlet boundary conditions for the domain, Ω. At this point, we introduce two separate scenarios motivated by the ideas behind domain decomposition presented in the introduction.

4.1. Continuous Coefficients (e.g., Constant)

The right-hand side of the model problem (11) is generated for the case where a is the identity matrix, b is zero, $r = 1$, and $u(x, y, z) = xyz(x - 2)(y - 1)(z - 1)(x^3 + y^3 + z^3)$. Thus, the coefficients for the

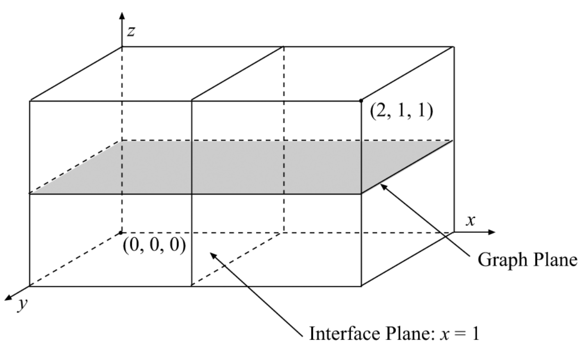

Figure 2. Points evaluated in error calculation are taken from the graph plane $\{(x, y, 1/2) | 0 \le x \le 2, 0 \le y \le 1\}$.

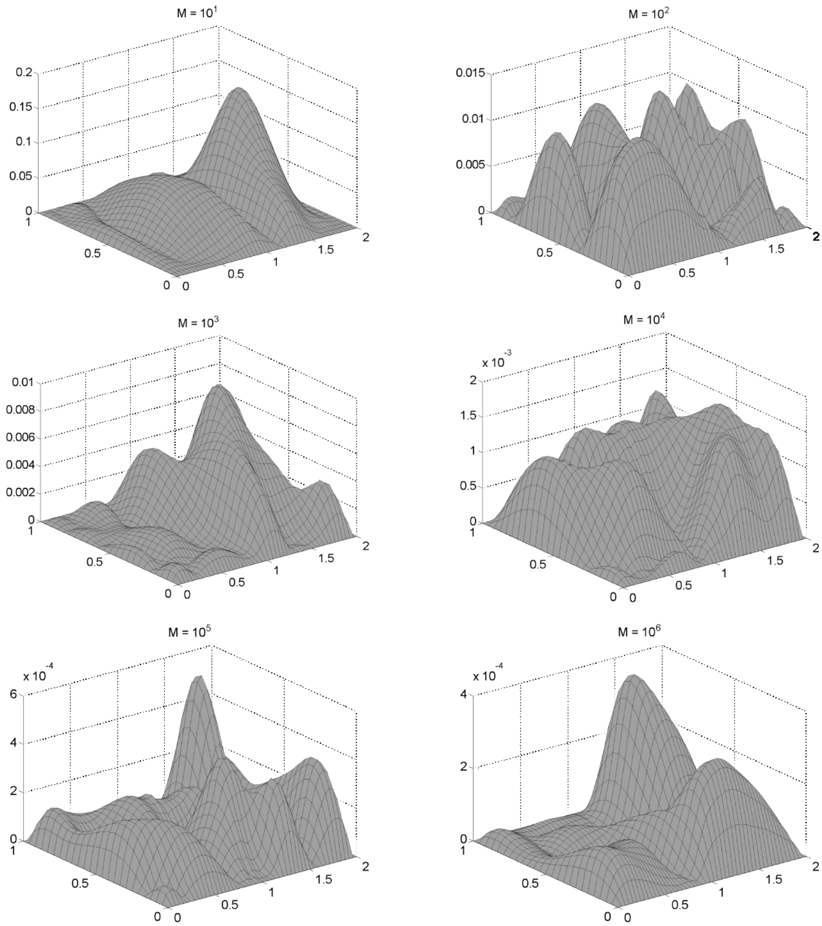

Figure 3. M-version for continuous a; point-wise difference $\left|E\left(\tilde{u}_N^{(j)}(\mathbf{x})\right) - u_N(\mathbf{x})\right|$ for $\mathbf{x} \in \{(x, y, 1/2)\}$ with $(p_1, p_2, q_1, q_2, r) = (4, 4, 3, 3, 3)$, $(m, n) = (4, 6)$, and $M = 10^k, k = 1, \ldots, 6$.

model problem are continuous. For this f, we then proceed to solve the discretized problem (14)–(16), using various scenarios.

First, we consider an M-version, where M refers to the number of paths, or realizations, generated in solving the stochastic version of our problem. For each path, a different set of values for η_{ijk} are chosen, creating a distinct solution. We consider the number of realizations $M = 10^k$ for $k = 1, 2, \ldots, 6$. In Table 1, we compare the resulting function $E\left(\tilde{u}_N^{(j)}\right)$ with the solution to the deterministic *finite element* problem, u_N, using the H^1 norm. Here we use m and n to represent the number of subdivisions in each spatial dimension for the subdomains, Ω_1 and

Table 2. s-version for continuous coefficients, a, where $\mathrm{Var}(\tilde{c})$ and $\mathrm{Var}_p(\tilde{c})$ are the calculated and predicted covariance matrix, respectively

	Number of realizations			
$\lvert s\rvert$	10	100	1000	10000
1/2	0.240190	0.236280	0.236056	0.235592
1/4	0.243740	0.231903	0.231101	0.230288
1/8	0.215798	0.125000	0.111440	0.102062
1/16	0.190141	0.092370	0.043397	0.049799
1/32	0.149923	0.076251	0.022725	0.013570

Ω_2, respectively. So, as a nonconforming case, consider $(m, n) = (4, 6)$, where the meshes in the subdomains do not coincide at the interface. We have also chosen $(p_1, p_2, q_1, q_2, r) = (4, 4, 3, 3, 3)$ (i.e., the orders of the polynomial spaces $V_1, V_2, \Lambda_1, \Lambda_2, T$). The last line of the table notes that the observed rate of convergence for each of the meshes is $O(M^{-1/2})$. Also, it is significant to note that, in this case, we have chosen the mesh for the stochastic integral approximation to be the same mesh used for the finite element spaces. Later we will consider cases in which they are chosen independently.

Some illustrations of M-version are presented in Figure 3, where we graph $\left|E\left(\tilde{u}_N^{(j)}\right) - u_N\right|$ over $\{(x, y, 1/2) \mid 0 \le x \le 2, 0 \le y \le 1\}$ (see Figure 2). Note that the difference here is between the approximated stochastic solution and the deterministic finite element solution.

Next we perform an experiment to confirm the results of Theorem 3.2, which we refer to as the s-version, in which s represents the partition $\{R_{ijk}\}$, with norm $|s|$ and with which we approximated the stochastic integral. Note that in the previous two examples, we assumed that $s = h$; that is, we used the same mesh for the stochastic integration as for the finite element space V. Let us now fix h and M and consider a

Table 3. Variance at a point for s-version for continuous coefficients, a

	Number of realizations, M				
$\lvert s\rvert$	10	100	1000	10000	$M \to \infty$
1/2	0.004111957	0.004405547	0.003732792	0.003690638	0.003791621
1/4	0.008337988	0.008661191	0.009626546	0.009643240	0.009626297
1/8	0.010475078	0.012738551	0.012825994	0.012427650	0.012565284
1/16	0.006650941	0.014864111	0.012623290	0.013584276	0.013369812
1/32	0.011587123	0.013217950	0.013861616	0.013468133	0.013588015

as $|s| \to 0$, $\mathrm{Var}(\tilde{u}_N(\mathbf{x})) \to 0.013663380$

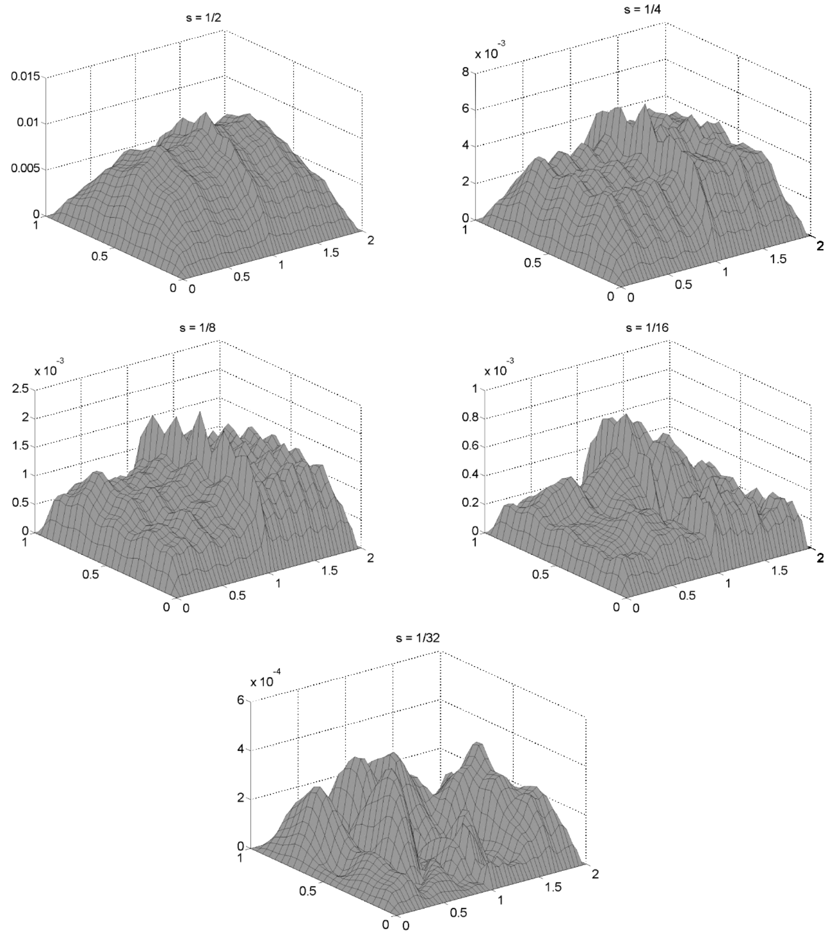

Figure 4. s-version for continuous a; point-wise difference $|\mathrm{Var}(\tilde{u}_h(\mathbf{x})) - \mathrm{Var}_p(\tilde{u}_N(\mathbf{x}))|$, where $\mathbf{x} \in \{(x, y, 1/2)\}$ with $(p_1, p_2, q_1, q_2, r) = (4, 4, 3, 3, 3)$ and $(m, n) = (4, 6)$, and $s = 1/2^k, k = 1, \ldots, 5$.

refinement of this mesh, i.e., $|s| \to 0$. We use the same parameters as in the M-version. That is, let us consider the case where $(m, n) = (4, 6)$ and $(p_1, p_2, q_1, q_2, r) = (4, 4, 3, 3, 3)$. We also fix $M = 10^4$. Now we choose s such that $|s| = 1/k$ for $k = 2, 4, 8, 16, 32$. Table 2 presents the maximum absolute difference between the calculated variance of the coefficients of $\tilde{u}_N$ and their exact variance as predicted by Theorem 3.2. We utilize the notation $\mathrm{Var}_p(\tilde{c}_i)$ for the predicted variance for the i-th coefficient of the stochastic solution. Note that for fixed $|s|$ and $M < \infty$, $\mathrm{Var}(\tilde{c}_i)$ will vary depending on the set of realizations. However, as M approaches infinity, the value of $\mathrm{Var}(\tilde{c}_i)$ will approach a constant value. As expected,

Table 4. h-version for continuous coefficients, a; demonstrates equation (24) when $(p_1, p_2, q_1, q_2, r) = (4, 4, 3, 3, 3)$ and $M = 10^4$

h	$\|E(\tilde{u}_N^{(j)}) - u\|_{1,\Omega}$	SUM	=	$\|E(\tilde{u}_N^{(j)}) - u_N^{(j)}\|_{1,\Omega}$	+	$\|u_N - u\|_{1,\Omega}$
1/2	0.0083667	0.0118854	=	0.0056628	+	0.0062226
1/4	0.0079890	0.0087152	=	0.0079528	+	0.0007624
1/6	0.0104510	0.0106800	=	0.0104484	+	0.0002316
1/8	0.0114909	0.0115898	=	0.0114903	+	0.0000995
1/10	0.0131293	0.0131808	=	0.0131292	+	0.0000516

the difference presented in this table appears to be converging at a rate of at least $O(|s|)$. It should be noted that in order for the convergence to be to zero, in general, we must also let $M \to \infty$.

Utilizing the results from Theorem 3.3 we can compare the variance at a fixed point $\mathbf{x} \in \Omega$. Let us consider the point $(1/2, 1/2, 1/2) \in \Omega$. In Table 3, the calculated variances of $\tilde{u}_N$ at this point are given for various choices of $|s|$ and M. As before, note that for fixed $|s|$ and $M < \infty$, $\mathrm{Var}(\tilde{u}_N(\mathbf{x}))$ will vary, but as M approaches infinity, it approaches a constant value. Figure 4 graphs the absolute difference between the calculated variance and the predict variance over the graph plane.

Finally, we consider an h-version, where we fix the number of realizations and allow the norm of the mesh to go to zero. Again, we present the nonconforming case. As before, let us choose $(p_1, p_2, q_1, q_2, r) = (4, 4, 3, 3, 3)$. We fix $M = 10^4$ and we select $(m, n) = (2k, 3k)$, $k = 1, 2, \ldots, 5$, for the refinement of our meshes. This h-version is analogous to the standard h-version considered for deterministic finite element methods. By simple application of the triangle inequality,

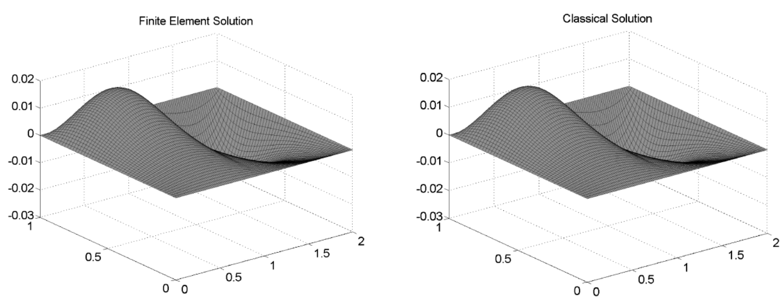

***Figure* 5.** H^1 error = 0.00000000, $(p_1, p_2, q_1, q_2, r) = (6, 6, 4, 4, 4)$. Deterministic nonconforming three-field FEM recovers exact solution containing discontinuous coefficients. Solution graphed on $\{(x, y, 1/2)\}$.

Table 5. Stochastic M-version for discontinuous coefficients, a, with $(p_1, p_2, q_1, q_2, r) = (4, 4, 3, 3, 3)$, $(m, n) = (4, 6)$, and $M = 10^k$, $k = 1, \ldots, 6$

M	$\left\|E(\tilde{u}_N^{(j)}) - u_N\right\|_{1,\Omega}$
10	0.14415623
100	0.06324116
1000	0.01593848
10000	0.00506422
100000	0.00180456
1000000	0.00053371
Rate	$O(M^{-1/2})$

we see that the error between the stochastic finite element solution and the classical solution is bounded as follows,

$$\left\|E\left(\tilde{u}_N^{(j)}\right) - u\right\|_{1,\Omega} \leq \left\|E\left(\tilde{u}_N^{(j)}\right) - u_N\right\|_{1,\Omega} + \|u_N - u\|_{1,\Omega}. \tag{24}$$

We have theoretical results for each of the terms of the right-hand side of (24). However, when the experiments were conducted with $M = 10^4$ fixed, the error, $\left\|E\left(\tilde{u}_N^{(j)}\right) - u\right\|_{1,\Omega}$ becomes dominated by the error $\left\|E\left(\tilde{u}_N^{(j)}\right) - u_N\right\|_{1,\Omega}$, for which we have no guarantee of convergence for fixed M and $h \to 0$. Table 4 gives each of the H^1 errors in (24). It is clear that this inequality holds in this table.

It is notable that, if we were to allow $M \to \infty$ and $|s| \to 0$, then the h-version would behave the same in both the deterministic and stochastic cases. The authors will explore details of this in a future paper. Additionally, in all three versions, we fixed the orders of the polynomial spaces. The case in which we allow these to vary will be considered in a future paper, as well.

4.2. Discontinuous Coefficients (e.g., Piecewise Constant)

We now consider the case in which a is discontinuous. This might arise in the cases where the domains, Ω_1 and Ω_2, represent different materials or distinct components of the model (e.g., the population model). In light of this, let us consider

$$a(\mathbf{x}) = \begin{cases} 1 & \text{for } \mathbf{x} \in \Omega_1 \\ 2 & \text{for } \mathbf{x} \in \Omega_2. \end{cases}$$

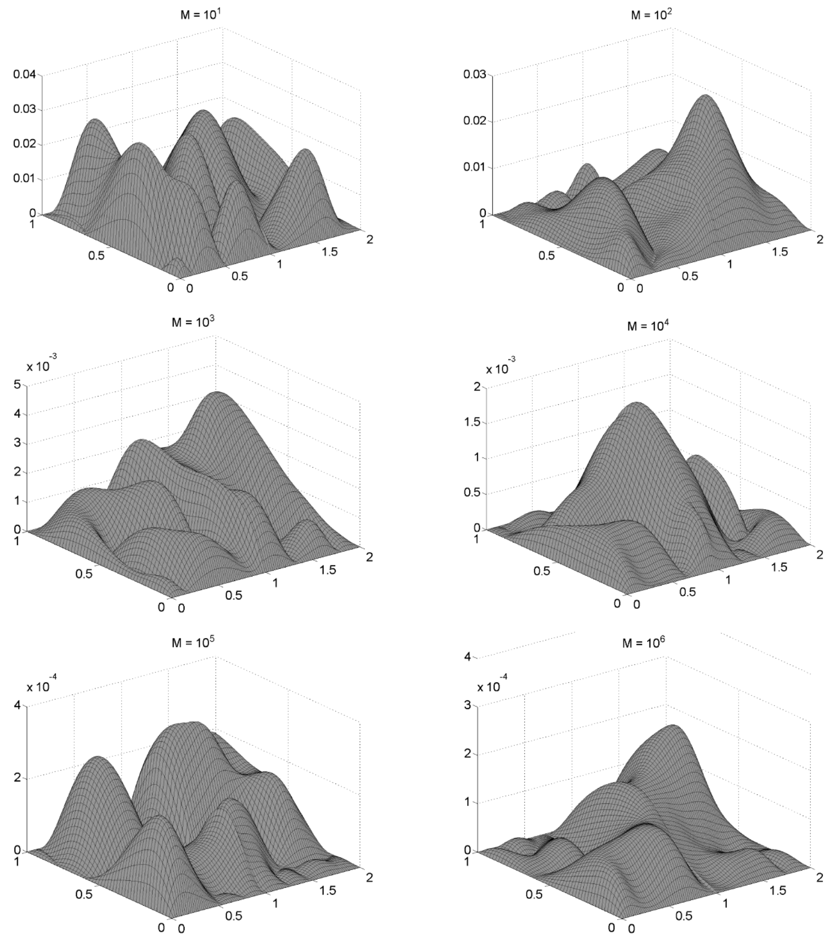

Figure 6. Stochastic M-version for discontinuous a. Point-wise difference $\left|E\left(\tilde{u}_N^{(j)}(\mathbf{x})\right) - u_N(\mathbf{x})\right|$ for $\mathbf{x} \in \{(x, y, 1/2)\}$ with $(p_1, p_2, q_1, q_2, r) = (4, 4, 3, 3, 3)$, $(m, n) = (4, 6)$, and $M = 10^k, k = 1, \ldots, 6$.

For simplicity, we continue to let $b = 0$ and

$$u(x, y, z) = \begin{cases} (x^2 - 2x)(y^2 - y)(z^2 - z)(x^3 - 1)(y^3 + z^3) & \text{for } \mathbf{x} \in \Omega_1 \\ (x^2 - 2x)(y^2 - y)(z^2 - z)(x^3 - 1)(y^3 + z^3)/2 & \text{for } \mathbf{x} \in \Omega_2. \end{cases}$$

This u was chosen specifically so that it maintained the properties that it is continuous in Ω and

$$a_1 \frac{\partial u_1}{\partial x} = a_2 \frac{\partial u_2}{\partial x},$$

where u_1, u_2 and a_1, a_2 represent the piecewise components of u and a, respectively. The discretized mixed formulation for the three-field method is able to recover the solution when the tensor product B-splines spaces, V^1 and V^2, are of degree 5 (see Figure 5).

Now we perform our numerical tests on this particular version of our problem. For brevity, we include only the results for the M-version, as the others produce similar results to the other scenario. We set $(m, n) = (4, 6)$ and $(p_1, p_2, q_1, q_2, r) = (4, 4, 3, 3, 3)$. We let $M = 10^k$ for $k = 1, 2, \ldots, 6$. Table 5 presents the absolute difference between the finite element solution to the deterministic version and the approximation to the stochastic version as M increases. We see clearly the error decreasing, as expected.

Figure 6 presents the point-wise difference along the same graph plane between the $E\left(\tilde{u}_N^{(j)}\right)$ and u_N.

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REFERENCES

1. Allen, E. J., S. J. Novosel, and Z. Zhang. 1998. Finite element and difference approximation of some linear stochastic partial differential equations. *Stochastics and Stochastics Reports* 64:117–142.
2. Babuška, I., R. Tempone, and G. E. Zouraris. 2004. Galerkin finite element approximations of stochastic elliptic partial differential equations. *SIAM Journal on Numerical Analysis* 42(2):800–825.
3. Bertoluzza, S. 2003. Analysis of a stabilized three-fields domain decomposition method. *Numer. Math.* 93(4):611–634.
4. Brenner, S. C., and L. R. Scott. 1994. *The Mathematical Theory of Finite Element Methods*. New York: Springer-Verlag.
5. Brezzi, F., and L. D. Marini. 1993. Macro hybrid elements and domain decomposition methods. In *Optimization et Côntrole*, eds. J. A. Désideri, L. Fezoui, B. Larrouturou, and B. Rousselet, Cépadués, Toulouse, 89–96.
6. Buckdahn, R., and E. Pardoux. 1990. Monotonicity methods for white noise driven quasi-linear SPDEs. In *Diffusion Processes and Related Problems in Analysis*, Vol. I, ed. M. Pinsky, Boston: Birkhaüser.
7. Cyganowski, S., P. Kloeden, and J. Ombach 2002. *From Elementary Probability to Stochastic Differential Equations with Maple.* Berlin, Heidelberg, New York: Springer-Verlag.
8. De Boor, C. 1978. *A Practical Guide to Splines.* New York: Springer.

9. Franklin, S. 2005. *A Computational Three-Field Methodology for Non-conforming Finite Elements over Partitioned Domains*. Ph.D. Dissertation, Texas Tech University.
10. Franklin, S., P. Seshaiyer, and P. W. Smith. 2004. A computational methodology to study coupled physical processes over partitioned domains. Submitted.
11. Higham, D. J. 2000. Mean-square an asymptotic stability of the stochastic theta method. *SIAM J. Numer Anal.* 38:753–769.
12. Kloeden, P., and E. Platen. 1992. *Numerical Solution of Stochastic Differential Equations*. New York: Springer-Verlag.
13. Manthey, R., and C. Stiewe. 1992. Existence and uniqueness of solutions to Volterra's population equation with diffusion and noise. *Stochastics and Stochastic Reports* 44:135–161.
14. Seshaiyer, P., and P. Smith. 2003. A non-conforming finite element method for sub-meshing. *Applied Mathematics and Computations* 139:85–100.
15. Platen, E. 1999. *An Introduction to Numerical Methods for Stochastic Differential Equations. Acta Numer. 8*, Cambridge: Cambridge University Press. 197–246.
16. Rohatgi, V. K. 1976. *An Introduction to Probability Theory and Mathematical Statistics*. New York: John Wiley & Sons, Inc., 264.
17. Sharp, W. D., and E. J. Allen. 2003. Development and analysis of quadrature and galerkin methods for approximate solution to the integral formulation of volterra's population equation with diffusion and noise. *International Journal of Pure and Applied Mathematics* 4:457–486.
18. Talay, D. 1988. Simulation and numerical analysis of stochastic differential systems. In *Effective Stochastic Analysis*, eds. P. Krée, and W. Wedig, Berlin: Springer-Verlag.
19. Walsh, J. B. 1986. An introduction to stochastic partial differential equations. *Lecture Notes in Mathematics 1180*, Berlin: Springer-Verlag, 265–439.