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OPTIMAL CONVERGENCE RATES OF *hp* MORTAR FINITE ELEMENT METHODS FOR SECOND-ORDER ELLIPTIC PROBLEMS *, **, ***

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Abstract. We present an improved, near-optimal hp error estimate for a non-conforming finite element method, called the mortar method (M0). We also present a new hp mortaring technique, called the mortar method (MP), and derive h, p and hp error estimates for it, in the presence of quasiuniform and non-quasiuniform meshes. Our theoretical results, augmented by the computational evidence we present, show that like (M0), (MP) is also a viable mortaring technique for the hp method.

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1. Introduction

The complexity of domains encountered in problems solved by the finite element method often necessitates the separate meshing of individual components or subdomains. The resulting submeshes are generally incompatible along interfaces, since it is quite cumbersome to impose conformity. A similar situation also arises when pre-existing meshes are to be used in a calculation, or when local refinement (say near singularities) is to be imposed in selected regions of a mesh.

Mortaring techniques are non-conforming methods that were developed to piece together such incompatible meshes (see [10,15]). The original paper [14] discussed two methods, the first being the "h–mortar finite element method" (which we denote in this paper as (M0)), where it was assumed that the meshes were quasiuniform but incompatible and the degree p was fixed (h version). In addition, a second method called the "mortar spectral element method" was also introduced (denoted here by (MP)). For this method the mesh was fixed, and only the degrees were allowed to vary. The non-conformity is due either to the incompatibility of polynomial degrees within each element, to the non-matching meshes, or to their combined effect. It was established that (M0)

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had optimal convergence properties with respect to the mesh size, and (MP) with respect to the degree. These results were generalized to three dimensions in [7]. See also [1–3, 8, 9], for related results.

In [25–27] the *hp* version of method (M0) was considered. The practical motivation was an implementation of mortaring techniques in the *hp* commercial code MSC-NASTRAN. The method (M0) was analyzed for the case that the meshes were non-quasiuniform (e.g. for *radical* and *geometrical* meshes, see [5]) and for the case that both the meshes and the degrees were allowed to vary. (Such *hp* techniques are essential if singularities are to be well-approximated). In particular, it was shown that: (1) the h version is optimal for non-quasiuniform meshes as long as a minor meshing condition is satisfied; (2) the p version satisfies an error estimate that is suboptimal by $O(p^{\frac{3}{4}})$; (3) the *hp* version with geometrical meshes gives exponential convergence.

Experimental results supported the above conclusions, except that the $O(p^{\frac{3}{4}})$ suboptimality was not observed. In [25, 28], two variations of (M0), called (M1) and (M2) were introduced, which had a simpler formulation than (M0), generalized more easily to three dimensions, and satisfied similar estimates.

Our goal in this paper is two-fold. First of all, we show that the $O(p^{\frac{3}{4}})$ suboptimality in [26, 27] may be removed, to be replaced by an $O(p^{\varepsilon})$ loss ($\varepsilon > 0$ arbitrary). Hence the method (M0) is essentially optimal in h *and* p.

Second, we extend the method (MP) to the case of incompatible meshes (possibly non-quasiuniform) where both h and p are allowed to vary, *i.e.* we consider it in the context of *hp* implementation. Our results show that (MP) is optimal in p even when the meshes are incompatible, and again yields exponential *hp* convergence with geometrical meshes. In terms of h, the method can suffer a loss of upto $O(h^{\frac{1}{2}})$ for solutions that are sufficiently smooth (the loss did not appear very significant in our numerical results in Section 6). Hence (MP) could be considered as another viable candidate for *hp* implementation, though in light of the optimality of (M0) proved here, the latter may be a better choice.

The plan of our paper is as follows. In Section 2, we describe the (M0) method, and establish the $O(p^{\varepsilon})$ optimality for this method. In Section 3, we introduce the method (MP), while Section 4 contains a proof of the *hp* error estimate for quasiuniform meshes. Section 4.1 extends this optimal result to the case of weakly regular solutions. In Section 5, we prove improved convergence results for non-quasiuniform meshes. Essentially, we show here that the "mortar projection operator" for (MP) is bounded as $O(p^{\varepsilon})$ in the appropriate norm, independently of the mesh (for (M0) it was shown that the corresponding projection operator is bounded as $O(p^{\frac{3}{4}+\varepsilon})$ provided the mesh was no more than geometrically refined). Finally, Section 6 contains the results of numerical experiments.

1.1. **Notations**

Let us assume that we have a Lipschitz domain $\mathcal{C} \subset \mathbb{R}^2$ and the generic point of $\mathcal{C}$ is denoted $\mathbf{x}$. The classical Lebesgue space of square integrable functions $\mathrm{L}^2(\mathcal{C})$ is endowed with the inner product:

$$(\varphi, \psi) = \int_{\mathcal{C}} \varphi\psi \ \mathrm{d}\mathbf{x}.$$

We assume standard Sobolev space notations, $\mathrm{H}^m(\mathcal{C})$, $m \geq 1$, provided with the norm:

$$\|\psi\|_{\mathrm{H}^m(\mathcal{C})} = \Big(\sum_{0\leq|\alpha|\leq m} \|\partial^\alpha \psi\|^2_{\mathrm{L}^2(\mathcal{C})} \Big)^{\frac{1}{2}},$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi–index in $\mathbb{N}^2$ and the symbol ∂^α represents a partial derivative. The fractional Sobolev space $\mathrm{H}^\tau(\mathcal{C}), \tau \in \mathbb{R}_+ \setminus \mathbb{N}$, is defined by its norm (see [22])

$$\|\psi\|_{\mathrm{H}^\tau(\mathcal{C})} = \left(\|\psi\|^2_{\mathrm{H}^m(\mathcal{C})} + \sum_{|\alpha|=m} \int_{\mathcal{C}} \int_{\mathcal{C}} \frac{(\partial^\alpha\psi(\mathbf{x}) - \partial^\alpha\psi(\mathbf{y}))^2}{|\mathbf{x}-\mathbf{y}|^{2+2\theta}} \ \mathrm{d}\mathbf{x} \ \mathrm{d}\mathbf{y} \right)^{\frac{1}{2}},$$

where $\tau = m+\theta$, m and $\theta \in]0,1[$ being the integer part and the fractional part of τ respectively. The closure in $\mathrm{H}^\tau(\mathcal{C})$ of the set $\mathscr{D}(\mathcal{C})$ of indefinitely differentiable functions whose support is contained in $\mathcal{C}$ is denoted $\mathrm{H}^\tau_0(\mathcal{C})$. We will also use the spaces $\mathrm{H}^\tau(\mathcal{J}), \mathrm{H}^\tau_0(\mathcal{J})$, where $\mathcal{J} \subset \mathbb{R}$.

For any portion of the boundary $\gamma \subset \partial\mathcal{C}$, the space $\mathrm{H}^{\frac{1}{2}}(\gamma)$ is the set of the traces over γ of all the functions of $\mathrm{H}^1(\mathcal{C})$ and $\mathrm{H}^{-\frac{1}{2}}(\gamma)$ is its topological dual space. The duality pairing between $\mathrm{H}^{-\frac{1}{2}}(\gamma)$ and $\mathrm{H}^{\frac{1}{2}}(\gamma)$ is $\langle\cdot,\cdot\rangle_{*,\gamma}$. The special space $\mathrm{H}^{\frac{1}{2}}_{00}(\gamma)$ is the subspace of $\mathrm{H}^{\frac{1}{2}}(\gamma)$ of the traces of all functions belonging to $\mathrm{H}^1_0(\mathcal{C},\gamma^c) = \big\{\psi \in \mathrm{H}^1(\mathcal{C}),\ \psi_{|\gamma^c} = 0\big\}$, where $\gamma^c = \partial\mathcal{C} \setminus \gamma$. It is endowed with the quotient norm,

$$\|\varphi\|_{\mathrm{H}^{\frac{1}{2}}_{00}(\gamma)} = \inf_{\psi\in \mathrm{H}^1_0(\mathcal{C},\gamma^c)} \|\psi\|_{\mathrm{H}^1(\mathcal{C})}.$$

1.2. **Model problem**

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ with boundary $\Gamma = \partial\Omega$ divided into Γ_D and Γ_N. Given $f \in \mathrm{L}^2(\Omega)$ and $g \in \mathrm{L}^2(\Gamma_N)$, consider the Poisson's problem with mixed boundary conditions: Find u such that,

$$\left\{\begin{array}{lll} -\Delta u & = f & \text{in } \Omega,\\ u & = 0 & \text{on } \Gamma_D,\\ \dfrac{\partial u}{\partial \mathbf{n}} & = g & \text{on } \Gamma_N. \end{array}\right. \tag{1.1}$$

The standard variational formulation which is based on a Green's integration formula for this problem becomes: Find $u \in \mathrm{H}^1_D(\Omega) = \mathrm{H}^1_0(\Omega,\Gamma_D)$ such that

$$\int_\Omega \nabla u \ .\ \nabla v \ \mathrm{d}\mathbf{x} = \int_\Omega f\ v\ \ \mathrm{d}\mathbf{x} + \int_{\Gamma_N} g\ v\ \mathrm{d}\Gamma \qquad \forall v \in \mathrm{H}^1_D(\Omega). \tag{1.2}$$

Problem (1.1) is well posed and has a unique solution $u \in \mathrm{H}^1_D(\Omega)$. Moreover, the following stability condition holds:

$$\|u\|_{\mathrm{H}^1(\Omega)} \le C(\|f\|_{\mathrm{L}^2(\Omega)} + \|g\|_{\mathrm{L}^2(\Gamma_N)}).$$

Remark 1.1. The hypothesis $g \in \mathrm{L}^2(\Gamma_N)$ is made only for the clarity of the presentation. The results however can be easily extended to the more general case where $g \in (\mathrm{H}^{\frac{1}{2}}_{00}(\Gamma_N))'$ without much difficulty.

2. The non-conforming finite element method (M0)

We assume that the domain Ω is broken arbitrarily into k^* non-overlapping polygonal subdomains $(\Omega_k)_{1\le k\le k^*}$. The edges of Ω_k are denoted by $(\Gamma_{k,j})_{1\le j\le j^*_k}$ and the vertices of each $\Gamma_{k,j}$ are denoted as $\nu^1_{k,j}$ and $\nu^2_{k,j}$. Such a restriction on the shape of the subdomains and the global domain Ω is made only for the sake of simplicity. Also, to avoid technical concerns we assume that the portion of $\partial\Omega_k$ contained in $\partial\Omega$ is a union of complete edges, and that the boundary condition does not change type in the interior of any edge of Ω_k. The outward normal on the whole boundary $\partial\Omega_k$ will be denoted as $\mathbf{n}_k$, when required. The decomposition is called *geometrically conforming* if $\Gamma_{k,j} \cap \Gamma_{\ell,i} \neq \emptyset$ implies either that the two edges are identical, or that the intersection contains a single common vertex. Otherwise, the decomposition is called *geometrically non-conforming*. Following the terminology of [14] the skeleton is defined to be

$$\mathcal{S} = \left(\bigcup_{k=1}^{k^*} \partial\Omega_k\right) \setminus \partial\Omega.$$

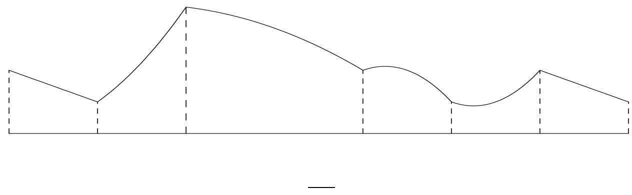

FIGURE 1. Example of the shape of the $M^\delta(\Gamma_{k,j})$-functions for $p_k = 2$.

Among several possible choices, we select a set of edges γ_m which we call *mortars* such that

$$\mathcal{S} = \bigcup_{m=1}^{m^*} \gamma_m, \qquad \gamma_m \cap \gamma_n = \emptyset \text{ if } m \neq n, \qquad \gamma_m = \Gamma_{k(m),j(m)}.$$

The mortar approximation of the Poisson problem we use is based on the p and hp finite elements (see [4–6, 20, 23, 24]). The elements used in the subdomain Ω_k are specified by the parameter $\delta_k = (h_k, p_k)$, where h_k goes to zero and/or p_k tends to infinity. For any k $(1 \leq k \leq k^*)$, let $\mathscr{T}_k^\delta$ be a partition of Ω_k into triangulations with a maximum size h_k. This mesh is assumed regular in the classical sense [18]. Since the $(\mathscr{T}_k^\delta)_k$ are generated independently, the meshes do not have to conform at the interface. For any $\kappa \in \mathscr{T}_k^\delta$ and any $r \in \mathbb{N}, \mathcal{P}_r(\kappa)$ denotes the set of polynomials of total degree $\leq r$. The local spaces are then chosen to be

$$X^\delta(\Omega_k) = \Big\{ v_k^\delta \in (\mathscr{C}(\overline{\Omega}_k))^2, \quad \forall \kappa \in \mathscr{T}_k^\delta, \ v_{k|\kappa}^\delta \in \mathcal{P}_{p_k}(\kappa), \quad v_{k|\partial\Omega_k \cap \Gamma_D}^\delta = 0 \Big\}.$$

We denote by $W^\delta(\Gamma_{k,j})$, the space of the traces on $\Gamma_{k,j}$ of all the functions of $X^\delta(\Omega_k)$. In particular, for any $m, 1 \leq m \leq m^*$, the local mortar space is $W^\delta(\gamma_m) = W^\delta(\Gamma_{k(m),j(m)})$ while the global mortar space is defined to be

$$W^\delta(\mathcal{S}) = \Big\{ \varphi^\delta = (\varphi_m^\delta)_m \in \mathscr{C}(\mathcal{S}), \quad \forall m, 1 \leq m \leq m^*, \qquad \varphi_m^\delta \in W^\delta(\gamma_m) \Big\}.$$

Given these tools, the approximation functions v^δ are taken locally in $X^\delta(\Omega_k)$ and glued together through the interfaces by some suitable matching conditions. First, we consider method (M0), investigated in [7, 10, 14, 26]. Each edge $\Gamma_{k,j}$ inherits from $\mathscr{T}_k^\delta$ a one–dimensional partition $t_{k,j}^\delta$ on which is built the space (see Fig. 1)

$$\begin{aligned} M^\delta(\Gamma_{k,j}) = \Big\{ \psi^\delta \in \mathscr{C}(\overline{\Gamma}_{k,j}), \quad & \forall t \in t_{k,j}^\delta, \ \psi_{|t}^\delta \in \mathcal{P}_{p_k}(t), \\ & \psi_{|t}^\delta \in \mathcal{P}_{p_k - 1}(t) \text{ if } t \text{ is the first or the last element of } \Gamma_{k,j} \Big\}. \end{aligned}$$

The global approximation space is given by

$$\begin{aligned} X^\delta(\Omega) = \Bigg\{ v^\delta = (v_k^\delta)_k \in \prod_{k=1}^{k^*} X^\delta(\Omega_k) \text{ such that: } & \exists \varphi^\delta \in W^\delta(\mathcal{S}), \forall k, j \} \\ & v_k^\delta(\nu) = \varphi^\delta(\nu), \qquad \text{for } \nu = \nu_{k,j}^1 \text{ or } \nu = \nu_{k,j}^2 \\ & \forall \psi^\delta \in M^\delta(\Gamma_{k,j}), \qquad \int_{\Gamma_{k,j}} (v_k^\delta - \varphi^\delta)\psi^\delta \, \mathrm{d}\Gamma = 0 \Bigg\}. \end{aligned}$$

Note that the Dirichlet boundary conditions are incorporated in $X^\delta(\Omega_k)$. Since it is not embedded in $\mathrm{H}_0^1(\Omega)$, the space $X^\delta(\Omega)$ is equipped with the Hilbertian broken norm,

$$\|v^\delta\|_* = \left(\sum_{k=1}^{k^*} \|v_k^\delta\|^2_{\mathrm{H}^1(\Omega_k)^2}\right)^{\frac{1}{2}}.$$

The discrete problem obtained by a Galerkin procedure then becomes: Find $u^\delta \in X^\delta(\Omega)$ satisfying,

$$a(u^\delta, v^\delta) = \int_\Omega f\, v^\delta\, \mathrm{d}\mathbf{x} + \int_{\Gamma_N} g\, v^\delta\, \mathrm{d}\Gamma, \qquad \forall v^\delta \in X^\delta(\Omega) < \tag{2.1}$$

where

$$a(u^\delta, v^\delta) = \sum_{k=1}^{k^*} \int_{\Omega_k} \nabla u_k^\delta \,.\, \nabla v_k^\delta\, \mathrm{d}\mathbf{x}.$$

The discretization is nonconforming, however, the discrete solution u^δ approximates the exact one u in an optimal way (see [14, 26]). More precisely, for fixed $(p_k)_k$ there exists a constant $C = C((p_k)_k)$ not depending on $(h_k)_k$ such that, if $u_k = u_{|\Omega_k} \in \mathrm{H}^{\tau_k}(\Omega_k)$, we have

$$\|u - u^\delta\|_* \leq C \sum_{k=1}^{k^*} h_k^{\eta_k - 1} |\log h_k|^{\frac{1}{2}} \|u_k\|_{\mathrm{H}^{\eta_k}(\Omega_k)}, \tag{2.2}$$

with $\eta_k = \min(\tau_k, p_k + 1)$. Under the same hypothesis, doing a full *hp* analysis gives the following result for the method (M0) (see [25, 26]),

$$\|u - u^\delta\|_* \leq C \sum_{k=1}^{k^*} \frac{h_k^{\eta_k - 1}}{p_k^{\tau_k - 1}} p_k^{\frac{3}{4}} |\log h_k|^{\frac{1}{2}} \|u_k\|_{\mathrm{H}^{\tau_k}(\Omega_k)}. \tag{2.3}$$

Below, we show how this estimate can be improved to replace the factor $p_k^{\frac{3}{4}}$ by p_k^ε.

Remark 2.1. Estimate (2.2) is proved in [12], it is a correction of that firstly given in [14] where $|\log h_k|^{\frac{1}{2}}$ was missing. This term appears when handling the consistency error when the decomposition is geometrically non-conforming and is removed otherwise.

Remark 2.2. In [25, 26], is also considered the case of *non-quasiuniform* meshes $\mathscr{T}_k^\delta$, where it is shown that the h-convergence remains *optimal* and that *exponential* convergence is obtained when the *hp* method is used over meshes geometrically refined near singularity points such as corners.

2.1. Improved *hp* estimate for (M0)

We use an interpolation argument to improve (2.3). For clarity alone, we take $g = 0$, but the result easily extends to this case as well.

Theorem 2.3. *Assume the exact solution $u \in \mathrm{H}_D^1(\Omega)$ of problem (1.1) is such that $u_k = u_{|\Omega_k} \in \mathrm{H}^{\tau_k}(\Omega_k), \tau_k \geq 1$. Then, for any $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that for u^δ, the solution of* (M0)*, with $g = 0$,*

$$\|u - u^\delta\|_* \leq C(\varepsilon) \sum_{k=1}^{k^*} \frac{h_k^{\eta_k - 1}}{p_k^{\tau_k - 1}} p_k^\varepsilon |\log h_k|^{\frac{1}{2}} \Big(\|u_k\|_{\mathrm{H}^{\tau_k}(\Omega_k)} + \|f_k\|_{\mathrm{L}^2(\Omega_k)}\Big). \tag{2.4}$$

where $\eta_k = \min(\tau_k, p_k + 1)$.

Proof. Let $\tau = (\tau_k)_k$ and ε be given. For simplicity, we prove the theorem where $\tau_k = \tau, h_k = h, p_k = p, \forall k (1 \leq k \leq k^*)$. Define the space $Y^\sigma(\Omega)$ by

$$Y^\sigma(\Omega) = \Big\{ v \in \mathrm{H}^1_D(\Omega), \quad v_k \in \mathrm{H}^\sigma(\Omega_k), \quad \Delta u \in \mathrm{L}^2(\Omega) \Big\}, \tag{2.5}$$

with the mesh-dependent norm,

$$\|u\|_{Y^\sigma(\Omega)} = \frac{h^{\xi-1}}{p^{(\sigma-1)\alpha}} |\log h|^{\frac{1}{2}} \sum_{k=1}^{k^*} \Big(\|u_k\|_{\mathrm{H}^\sigma(\Omega_k)} + \|\Delta u\|_{\mathrm{L}^2(\Omega_k)} \Big). \tag{2.6}$$

Here $\alpha = 1 - \frac{\varepsilon}{\tau - 1}$ and $\xi = \min(\sigma, p+1)$. By Theorem 3.1 of [11], we have the following interpolation result

$$\Big[Y^\sigma(\Omega), Y^1(\Omega) \Big]_{1-\theta} = Y^{\theta(\sigma-1)+1}(\Omega). \tag{2.7}$$

Then, consider the linear operator $T : u \mapsto u^\delta$. The stability condition says that T is continuous from $Y^1(\Omega)$ into $\prod_{k=1}^{k^*} H^1_D(\Omega_k)$ $(-\Delta u_k = f_k$ and $g = 0)$,

$$\|u - T(u)\|_* \leq C \sum_{k=1}^{k^*} (\|u_k\|_{\mathrm{H}^1(\Omega_1)} + \|\Delta u_k\|_{\mathrm{L}^2(\Omega_k)}) \leq C \|u\|_{Y^1(\Omega)}. \tag{2.8}$$

Next, choose τ such that

$$\frac{3}{4} \leq \frac{\sigma - 1}{\tau - 1}\, \varepsilon, \quad i.e. \quad \sigma \geq \frac{3(\tau-1)}{4\varepsilon} + 1.$$

Then

$$p^{-(\sigma-1)\alpha} = p^{-(\sigma-1)+\varepsilon\frac{\sigma-1}{\tau-1}} \geq p^{-(\sigma-1)+\frac{3}{4}},$$

so that by (2.3), we have

$$\|u - T(u)\|_* \leq C \|u\|_{Y^\sigma(\Omega)}. \tag{2.9}$$

Now let $\theta = \frac{\tau-1}{\sigma-1}$, so that $1 + \theta(\sigma - 1) = \tau$. Interpolating (2.8), (2.9), we have with $\eta = \min(\tau, p+1)$,

$$\begin{aligned} \|u - T(u)\|_* &\leq C \|u\|_{[Y^\sigma(\Omega), Y^1(\Omega)]_{1-\theta}} \leq C \|u\|_{Y^\tau(\Omega)} \\ &\leq C \frac{h^{\eta-1}}{p^{(\tau-1)\alpha}} |\log h|^{\frac{1}{2}} \sum_{k=1}^{k^*} \Big(\|u_k\|_{\mathrm{H}^\tau(\Omega_k)} + \|\Delta u\|_{\mathrm{L}^2(\Omega_k)} \Big). \end{aligned}$$

But $(\tau - 1)\alpha = (\tau - 1) - \varepsilon$, which gives the result. □

Remark 2.4. Estimate (2.4) can be sharpened further. Indeed, it is readily proven that the log-extra-term is actually $|\log h_k|^{\frac{\tau-1}{2(p-1)}}$, so that for large values for p this term has not significant effect on the numerical experiences.

3. The hp mortar finite element method (MP)

We now define a different mortar method (MP), which enforces a different hp mortar matching condition as follows. On each face $\Gamma_{k,j}$, that is not a mortar, a projection operator $\pi^\delta_{k,j}$, which is more specific to the p version is defined. For clarity, we will study its properties on the reference segment $A = (-1, 1)$ and so the projection will simply be denoted π^δ, where $\delta = (h, p)$. Similar results can be deduced on $\pi^\delta_{k,j}$ in a straightforward way. Let t^δ be the triangulation of A characterized by the subdivision $(\xi_i)_{0\le i\le i^*}$ with $\xi_0 = -1$ and $\xi_{i*} = 1$ for the vertices ν^1 and ν^2 respectively and $(t_i =]\xi_i, \xi_{i+1}[)_{0\le i\le i^*-1}$ for its elements. The space $W^\delta(A)$ is then defined as,

$$W^\delta(A) = \Big\{\chi^\delta \in \mathscr{C}(\overline{A}), \quad \forall t \in t^\delta,\ \chi^\delta_{|t} \in \mathcal{P}_p(t)\Big\}.$$

Next, we consider the operator

$$\pi^\delta : \mathrm{H}^1(A) \to W^\delta(A)$$

such that: $\forall \chi \in \mathrm{H}^1(A)$,

$$\forall i (0 \le i \le i^*), \qquad \pi^\delta\chi(\xi_i) = \chi(\xi_i), \tag{3.1}$$

$$\forall \psi^\delta \in \mathcal{P}_{p-2}(t), \qquad \int_{t_i} (\chi - \pi^\delta\chi)\psi^\delta \,\mathrm{d}A = 0. \tag{3.2}$$

Then, we have the following two technical lemmas.

Lemma 3.1. *The operator π^δ is such that: $\forall \chi \in \mathrm{H}^1(A)$,*

$$\int_A (\chi - \pi^\delta\chi)'(\psi^\delta)' \,\mathrm{d}A = 0, \qquad \forall \psi^\delta \in W^\delta(A).$$

Proof. Writing

$$\int_A (\chi - \pi^\delta\chi)'(\psi^\delta)' \,\mathrm{d}A = \sum_{i=0}^{i^*-1} \int_{t_i} (\chi - \pi^\delta\chi)'(\psi^\delta)' \,\mathrm{d}A,$$

and integrating by parts, we have,

$$\int_A (\chi - \pi^\delta\chi)'(\psi^\delta)' \,\mathrm{d}A = \sum_{i=0}^{i^*-1} [(\chi - \pi^\delta\chi)(\psi^\delta)']_{\xi_i}^{\xi_{i+1}} - \int_{t_i} (\chi - \pi^\delta\chi)(\psi^\delta)'' \,\mathrm{d}A.$$

The lemma follows by observing the fact that $(\psi^\delta_{|t_i})'' \in \mathcal{P}_{p-2}(t)$. $\square$

Lemma 3.2. *For any $\tau \ge 1$, the following estimates hold: $\forall \chi \in \mathrm{H}^\tau(A)$,*

$$(\frac{h}{p})^{-1}\|\chi - \pi^\delta\chi\|_{\mathrm{L}^2(A)} + \|(\chi - \pi^\delta\chi)'\|_{\mathrm{L}^2(A)} \le C\frac{h^{\eta-1}}{p^{\tau-1}}\|\chi\|_{\mathrm{H}^\tau(A)}, \tag{3.3}$$

$$\|\chi - \pi^\delta\chi\|_{\mathrm{H}^{\frac12}_{00}(A)} \le C\frac{h^{\eta-\frac12}}{p^{\tau-\frac12}}\|\chi\|_{\mathrm{H}^\tau(A)}. \tag{3.4}$$

where $\eta = \min(\tau, p+1)$.

Proof. By Lemma (3.1) we obtain,

$$\|(\chi - \pi^\delta \chi)'\|_{\mathrm{L}^2(A)} \le \inf_{\psi^\delta \in W^\delta(A)} \|(\chi - \psi^\delta)'\|_{\mathrm{L}^2(A)},$$

From the *hp* finite element we deduce,

$$\|(\chi - \pi^\delta \chi)'\|_{\mathrm{L}^2(A)} \le C \frac{h^{\eta-1}}{p^{\tau-1}} \|\chi\|_{\mathrm{H}^\tau(A)}.$$

The L^2–estimate in 3.3 is then derived by the Aubin-Nitsche duality argument.

The $\mathrm{H}^{\frac12}_{00}$ estimate in 3.4 is proved by using a Hilbertian interpolation argument of the L^2 and H^1 estimates. Actually we have local estimates: $\forall i (0 \le i \le i^*)$,

$$\|\chi - \pi^\delta \chi\|_{\mathrm{H}^{\frac12}_{00}(t_i)} \le C \frac{h^{\eta-\frac12}}{p^{\tau-\frac12}} \|\chi\|_{\mathrm{H}^\tau(t_i)}.$$

□

We now revisit the mortar discretization and use all the ingredients to define the new approximation space,

$$\tilde{X}^\delta(\Omega) = \Big\{ v^\delta = (v_k^\delta)_k \in \prod_{k=1}^{k^*} X^\delta(\Omega_k) \quad \text{such that: } \exists \varphi^\delta \in W^\delta(\mathcal{S}),$$
$$\forall k (1 \le k \le k^*), \forall j (1 \le j \le j_k^*), \qquad v^\delta_{k|\Gamma_{k,j}} = \pi^\delta_{k,j} \varphi^\delta \Big\}.$$

It is clear that the stability of the mortar method (MP) is maintained, meaning that any function v^δ is associated with only one mortar function:

$$\forall m (1 \le m \le m^*), \qquad \varphi^\delta_m = v^\delta_{k(m)|\gamma_m}.$$

The discrete problem (MP) is then formulated in the same terms as in (2.1) where $X^\delta(\Omega)$ is replaced by $\tilde{X}^\delta(\Omega)$. This problem is well posed and has a unique solution $u^\delta \in \tilde{X}^\delta(\Omega)$ that satisfies

$$\|u^\delta\|_* \le C \left(\|f\|_{\mathrm{L}^2(\Omega)} + \|g\|_{\mathrm{L}^2(\Gamma_N)} \right).$$

It remains to carry out the numerical analysis of our technique.

4. *hp* Error estimate for (MP)

We need the Berg-Scott-Strang Lemma (see [16]) better known as the second Strang Lemma currently used for non conforming methods (see [18, 29]).

Lemma 4.1. *We have the following error estimate for u^δ, the discrete solution computed by the* (MP)*-mortar method,*

$$\|u - u^\delta\|_* \le C \left\{ \inf_{v^\delta \in \tilde{X}^\delta(\Omega)} \|u - v^\delta\|_* + \sup_{w^\delta \in \tilde{X}^\delta(\Omega)} \frac{1}{\|w^\delta\|_*} \Big(\sum_{k=1}^{k^*} \left\langle \frac{\partial u}{\partial \mathbf{n}_k}, w_k^\delta \right\rangle_{*,\partial\Omega_k} - \int_{\Gamma_N} g w^\delta \, \mathrm{d}\Gamma \Big) \right\}.$$

The evaluation of the consistency error should be done with a little care, even though we follow the methodology developed in [14].

Lemma 4.2. *Assume* $u_k = u_{|\Omega_k} \in \mathrm{H}^{\tau_k}(\Omega_k), \tau_k > \frac{3}{2}$, *then*

$$\sup_{w^\delta \in \tilde{X}^\delta(\Omega)} \frac{1}{\|w^\delta\|_*} \left(\sum_{k=1}^{k^*} \left\langle \frac{\partial u}{\partial \mathbf{n}_k}, w_k^\delta \right\rangle_{*,\partial\Omega_k} - \int_{\Gamma_N} g w^\delta \, \mathrm{d}\Gamma \right) \leq C \sum_{k=1}^{k^*} \frac{h_k^{\eta_k - 1}}{p_k^{\tau_k - 1}} \left| \log \frac{p_k}{h_k} \right|^{\frac{1}{2}} \|u_k\|_{\mathrm{H}^{\tau_k}(\Omega_k)},$$

with $\eta_k = \min(\tau_k, p_k + \frac{1}{2})$.

Proof. Only for simplicity we examine the case where $(h_k, p_k) = (h, p), \forall k (1 \leq k \leq k^*)$. Since $\frac{\partial u}{\partial \mathbf{n}_k}|_{\partial\Omega_k} \in \mathrm{L}^2(\partial\Omega_k)$ we have: $\forall w^\delta \in \tilde{X}^\delta(\Omega)$,

$$\sum_{k=1}^{k^*} \left\langle \frac{\partial u}{\partial \mathbf{n}_k}, w_k^\delta \right\rangle_{*,\partial\Omega_k} - \int_{\Gamma_N} g w^\delta \, \mathrm{d}\Gamma = \sum_{(k,j) \neq (k(m), j(m))} \int_{\Gamma_{k,j}} \frac{\partial u}{\partial \mathbf{n}_k} \left(w_k^\delta - \varphi^\delta \right) \, \mathrm{d}\Gamma$$

where φ^δ is the mortar function associated to w^δ and $(k,j) \neq (k(m), j(m))$ indicates that $\Gamma_{k,j}$ is not a mortar. Assuming the matching conditions we derive: $\forall \psi^\delta \in \mathrm{L}^2(\Gamma_{k,j})$ such that $\psi_{|t} \in \mathcal{P}_{p_k - 2}(t), \forall t \in t_{k,j}^\delta$,

$$\begin{aligned} \sum_{k=1}^{k^*} \left\langle \frac{\partial u}{\partial \mathbf{n}_k}, w_k^\delta \right\rangle_{*,\partial\Omega_k} - \int_{\Gamma_N} g w^\delta \, \mathrm{d}\Gamma &= \sum_{(k,j) \neq (k(m), j(m))} \int_{\Gamma_{k,j}} \left(\frac{\partial u}{\partial \mathbf{n}_k} - \psi^\delta \right) \left(w_k^\delta - \varphi^\delta \right) \, \mathrm{d}\Gamma, \\ &\leq \sum_{(k,j) \neq (k(m), j(m))} \left\| \frac{\partial u}{\partial \mathbf{n}_k} - \psi^\delta \right\|_{(\mathrm{H}^{\frac{1}{2}-\varepsilon}(\Gamma_{k,j}))'} \|w_k^\delta - \varphi^\delta\|_{\mathrm{H}^{\frac{1}{2}-\varepsilon}(\Gamma_{k,j})}. \end{aligned} \tag{4.1}$$

Then, as $\dfrac{\partial u}{\partial \mathbf{n}_k}|_{\Gamma_{k,j}} \in \mathrm{H}^{\tau_k - \frac{3}{2}}(\Gamma_{k,j})$, we have

$$\inf_{\psi^\delta; \psi^\delta|_t \in \mathcal{P}_{p-2}(t)} \left\| \frac{\partial u}{\partial \mathbf{n}_k} - \psi^\delta \right\|_{(\mathrm{H}^{\frac{1}{2}-\varepsilon}(\Gamma_{k,j}))'} \leq C \frac{h^{\eta_k - 1}}{p^{\tau_k - 1}} \frac{h^{-\varepsilon}}{p^{-\varepsilon}} \left\| \frac{\partial u}{\partial \mathbf{n}_k} \right\|_{\mathrm{H}^{\tau_k - \frac{3}{2}}(\Gamma_{k,j})}$$

with $\eta_k = \min(\tau_k, p_k + \frac{1}{2})$. Substituting this in (4.1) we get,

$$\begin{aligned} &\sum_{k=1}^{k^*} \left\langle \frac{\partial u}{\partial \mathbf{n}_k}, \; w_k^\delta \right\rangle_{*,\partial\Omega_k} - \int_{\Gamma_N} g w^\delta \, \mathrm{d}\Gamma \\ &\quad \leq C \sum_{(k,j) \neq (k(m), j(m))} \frac{h^{\eta_k - 1}}{p^{\tau_k - 1}} \frac{h^{-\varepsilon}}{p^{-\varepsilon}} \left\| \frac{\partial u}{\partial \mathbf{n}_k} \right\|_{\mathrm{H}^{\tau_k - \frac{3}{2}}(\Gamma_{k,j})} \left\| w_k^\delta - \varphi^\delta \right\|_{\mathrm{H}^{\frac{1}{2}-\varepsilon}(\Gamma_{k,j})}. \end{aligned} \tag{4.2}$$

Proceeding as in [12], we observe that

$$\begin{aligned} \sum_{(k,j) \neq (k(m), j(m))} \|w_k^\delta - \varphi^\delta\|_{\mathrm{H}^{\frac{1}{2}-\varepsilon}(\Gamma_{k,j})} &\leq \frac{C}{\sqrt{\varepsilon}} \|w^\delta\|_* \\ \left\| \frac{\partial u}{\partial \mathbf{n}_k} \right\|_{\mathrm{H}^{\tau_k - \frac{3}{2}}(\Gamma_{k,j})} &\leq C \|u_k\|_{\mathrm{H}^{\tau_k}(\Omega_k)}. \end{aligned}$$

Inserting this in (4.2) and choosing $\varepsilon = (\log \frac{p}{h})^{-1}$ completes the proof. □

Remark 4.3. As in the case of estimates for method (M0), the log term in Lemma 4.2 can be eliminated for the case that the subdomains form a geometrically conforming partition (see [12]). This also holds for all log terms present in estimates in the sequel.

Remark 4.4. The limitation $\eta_k \leq p_k + \frac{1}{2}$ is intrinsically connected to the h version finite element approximation results (in (4.1) the degree of $\psi^\delta|_t$ is at most $p_k - 2$). However, for concrete situations the exact solution is often singular so that for p large enough (as is often the case in practice), we will have $\tau_k \leq p_k + \frac{1}{2}$, so that no effective deterioration will occur. Even when the solution is smooth, we observe that the deterioration may not be very apparent, as observed from the computations in Section 6.

We are left with the best approximation error which turns out to be optimal. The proof of such results requires us to proceed as in [14].

Lemma 4.5. *Assume* $u_k = u_{|\Omega_k} \in \mathrm{H}^{\tau_k}(\Omega_k), \tau_k > \frac{3}{2}$, *then we have*

$$\inf_{v^\delta \in \tilde{X}^\delta(\Omega)} \|u - v^\delta\|_* \leq C \sum_{k=1}^{k^*} \frac{h_k^{\eta_k - 1}}{p_k^{\tau_k - 1}} \|u_k\|_{\mathrm{H}^{\tau_k}(\Omega_k)},$$

with $\eta_k = \min(\tau_k, p_k + 1)$.

Proof. The proof takes two steps, we build up a mortar function $\varphi^\delta \in W^\delta(\mathcal{S})$ that is close to $u_{|\mathcal{S}}$, then we construct v^δ associated with φ^δ that satisfies the expected error estimate.

i. For any $m(1 \leq m \leq m^*)$ let us define $\varphi_m^\delta = \pi_{k(m),j(m)}^\delta(u_{|\gamma_m})$ and set $\varphi^\delta = (\varphi_m^\delta)_m$. It is clear that φ^δ is continuous and belongs to $W^\delta(\mathcal{S})$. Moreover, by Lemma 3.2 the following holds

$$\sum_{m=1}^{m^*} \|u - \varphi^\delta\|_{\mathrm{H}_{00}^{\frac{1}{2}}(\gamma_m)} \leq C \sum_{k=1}^{k^*} \frac{h_k^{\eta_k - 1}}{p_k^{\tau_k - 1}} \|u_k\|_{\mathrm{H}^{\tau_k}(\Omega_k)}.$$

ii . For any $k(1 \leq k \leq k^*)$ there exists $w^\delta = (w_k^\delta)_k$ with $w_k^\delta \in X^\delta(\Omega_k)$ satisfying (see [4]),

$$\|u - v^\delta\|_{\mathrm{H}^1(\Omega_k)} \leq C \frac{h_k^{\eta_k - 1}}{p_k^{\tau_k - 1}} \|u_k\|_{\mathrm{H}^{\tau_k}(\Omega_k)}.$$

Then, we need to modify w^δ in order to enforce the matching conditions. It is realized as follows

$$v_k^\delta = w_k^\delta + r_k^\delta,$$

where the correcting term $r_k^\delta \in X^\delta(\Omega_k)$ and: $\forall j(1 \leq j \leq j_k^*)$,

$$r_{k|\Gamma_{k,j}}^\delta = (\pi_{k,j}^\delta \varphi^\delta - w_k^\delta)_{|\Gamma_{k,j}}.$$

It remains to prove that

$$\|r^\delta\|_* \leq C \sum_{k=1}^{k^*} \frac{h_k^{\eta_k - 1}}{p_k^{\tau_k - 1}} \|u_k\|_{\mathrm{H}^{\tau_k}(\Omega_k)},$$

which is done following the same lines as in [14] and using estimate (3.4). □

Putting together Lemmas 4.1, 4.2 and 4.5 produces the concluding result of the final convergence rate.

Theorem 4.6. *Assume the exact solution* $u \in \mathrm{H}_D^1(\Omega)$ *of problem (1.1) is such that* $u_k = u_{|\Omega_k} \in \mathrm{H}^{\tau_k}(\Omega_k)$, $\tau_k \geq \frac{3}{2}$. *Then, the discrete solution* $u^\delta \in \tilde{X}^\delta(\Omega)$ *computed by the hp-version of the* (MP)*–mortar finite element verifies the following error estimate*

$$\|u - u^\delta\|_* \leq C \sum_{k=1}^{k^*} \frac{h_k^{\eta_k - 1}}{p_k^{\tau_k - 1}} \left|\log \frac{p_k}{h_k}\right|^{\frac{1}{2}} \|u_k\|_{\mathrm{H}^{\tau_k}(\Omega_k)}, \tag{4.3}$$

with $\eta_k = \min(\tau_k, p_k + \frac{1}{2})$.

Remark 4.7. The continuity matching at the vertices of the elements of $\partial\Omega_k$ can be relaxed as in [7] and [8], and the global error estimate is preserved. As shown in [9] (see also [8]) this is very advisable when carrying computations on parallel machines, as the communication time between different processors is reduced.

Remark 4.8. When, $p_k = 1$, corresponding to the h-finite elements, we recover the convergence rate $\mathcal{O}(\sqrt{h_k})$ of the interpolation matching analyzed in [10]. Of course, in such a case it is better to use the mortar space $X^\delta(\Omega)$. In fact, in the regions where the h–version is more efficient, we can use the matching conditions of $X^\delta(\Omega)$, and if for some reason the p–version should be privileged we use those specified in $\tilde{X}^\delta(\Omega)$.

Remark 4.9. For the p–version of the finite element method estimate (4.2) becomes

$$\|u - u^\delta\|_* \leq C \sum_{k=1}^{k^*} \frac{|\log p_k|^{\frac{1}{2}}}{p_k^{\tau_k - 1}} \|u_k\|_{\mathrm{H}^{\tau_k}(\Omega_k)},$$

and is then optimal (up to $|\log p_k|^{\frac{1}{2}}$) as for the mortar spectral element method (see [12]).

4.1. Convergence results for weakly regular solutions

When the regularity exponent $(\tau_k)_k$ is lower than or equal to $\frac{3}{2}$, deriving the error estimate requires some more technical work as it resorts to Hilbertian interpolation argument. We need to assume that the domain Ω is star-shaped with respect to a ball. For clarity alone, we take $g = 0$.

Theorem 4.10. *Assume the exact solution $u \in \mathrm{H}^1_D(\Omega)$ of problem (1.1) is such that $u_k = u_{|\Omega_k} \in \mathrm{H}^{\tau_k}(\Omega_k)$, $\tau_k \leq \frac{3}{2}$. Then, the following estimate holds for method* (MP)

$$\|u - u^\delta\|_* \leq C \sum_{k=1}^{k^*} \frac{h_k^{\tau_k - 1}}{p_k^{\tau_k - 1}} \left|\log \frac{p_k}{h_k}\right|^{\frac{1}{2}} (\|u_k\|_{\mathrm{H}^{\tau_k}(\Omega_k)} + \|f_k\|_{\mathrm{L}^2(\Omega_k)}). \tag{4.4}$$

We leave the proof to the reader as it is made using a Hilbertian interpolation argument exactly as in Theorem 2.3.

5. Improved convergence rates for non-quasiuniform meshes

So far, the analysis carried out, though still valid for non quasi-uniform meshes, does not provide sharp estimates for some interesting categories of such meshes. Indeed, when the domain has corners, or has points where the type of boundary condition changes, the solution will possess r^α type singularities at such points (r being the distance to the corner under consideration). To deal with such singularities, highly graded meshes should be used in each domain in the vicinity of such points. For instance, *radical* meshes are the optimal ones to use in the h version, while *geometrical* meshes are optimal for the hp version (see e.g. [5]). To be a suitable candidate for hp implementation, the mortaring technique should be robust when such meshes lie along the interface. We therefore now consider these kind meshes for the (MP)-method.

Accordingly, let us consider the mortar projection π^δ on $A = (-1, 1)$ again, as defined by (3.1)-(3.2), but with the triangulation t^δ on A now being non-quasiuniform. (The degree p is assumed uniform on t^δ). Then the following stability estimate on π^δ is the key result of this section (compare with (3.4)).

Proposition 5.1. *Given $\varepsilon > 0$, there exists $C(\varepsilon)$ independent of p such that: $\forall \chi \in \mathrm{H}_0^{\frac{1}{2}+\varepsilon}(A)$,*

$$\|\pi^\delta \chi\|_{\mathrm{H}^{\frac{1}{2}}_{00}(A)} \leq C(\varepsilon) p^{\frac{1}{4}} \|\chi\|_{\mathrm{H}^{\frac{1}{2}+\varepsilon}(A)}.$$

The proof of Proposition 5.1 requires some preliminary technical lemmas. Let us consider the operator $\pi^\delta_{t_i} = \pi^\delta_{|t_i}$, it coincides with π^δ when the mesh t^δ is reduced to only one element.

Lemma 5.2. *The following holds:* $\forall\chi \in \mathrm{H}_0^{\frac{1}{2}+\varepsilon}(t_i)$

$$\|\pi^\delta_{t_i}\chi\|_{\mathrm{H}^{\frac{1}{2}}_{00}(t_i)} \le Cp^{\frac{1}{4}}|\chi|_{\mathrm{H}^{\frac{1}{2}+\varepsilon}(t_i)}. \tag{5.1}$$

The constant C is independant of t_i.

Proof. By scaling we can consider $t_i = (-1,1)$ and $\pi^\delta_{t_i}$ is then denoted by π^p. The operator π^p satisfies the following stability inequalities (see [13], Lemma A.1): $\forall\varphi \in \mathscr{D}(-1,1)$

$$\|\pi^p\chi\|_{\mathrm{L}^2(-1,1)} \le Cp^{\frac{1}{2}}\|\chi\|_{\mathrm{L}^2(-1,1)}$$
$$\|\pi^p\chi\|_{\mathrm{H}^1(-1,1)} \le C|\chi|_{\mathrm{H}^1(-1,1)}$$

Using density, Hilbertian interpolation arguments and observing that $\mathrm{H}^{\frac{1}{2}}_{00}(-1,1) \subset \mathrm{H}^{\frac{1}{2}+\varepsilon}(-1,1)$ completes the proof (actually back to t_i we should have a bound like $Cp^{\frac{1}{4}}h_i^\varepsilon$). □

Lemma 5.3. *Let $\varphi \in \mathrm{H}_0^{\frac{1}{2}+\varepsilon}(A)$ and let $\overline{\varphi}^h$ denote its piecewise linear interpolant on t^δ. Then for any $0 < \varepsilon < \frac{1}{2}$, there exists a constant $C(\varepsilon)$ such that*

$$\|\overline{\varphi}^h\|_{\mathrm{H}^{\frac{1}{2}+\epsilon}(A)} \le C(\varepsilon)\|\varphi\|_{\mathrm{H}^{\frac{1}{2}+\epsilon}(A)}. \tag{5.2}$$

Proof. We construct the square $Q = (-2,2)\times(-2,2)$ with $A = (-1,1)$ imbedded along the side $y = -2$. Let $\mathscr{T}^\delta(Q)$ denote a regular mesh on Q whose trace on A is t^δ. Extend φ by a norm-preserving operator to φ_e defined on ∂Q such that

$$\|\varphi_e\|_{\mathrm{H}^{\frac{1}{2}+\varepsilon}(\partial Q)} \le C\|\varphi\|_{\mathrm{H}^{\frac{1}{2}+\varepsilon}(A)},$$

and let $\Phi \in \mathrm{H}^{1+\varepsilon}(Q)$ be a stable extension of $\varphi_e \in \mathrm{H}^{\frac{1}{2}+\varepsilon}(\partial Q)$, satisfying

$$\|\Phi\|_{\mathrm{H}^{1+\epsilon}(Q)} \le C\|\varphi_e\|_{\mathrm{H}^{\frac{1}{2}+\epsilon}(\partial Q)} \le C\|\varphi\|_{\mathrm{H}^{\frac{1}{2}+\varepsilon}(A)}. \tag{5.3}$$

Define Φ^h to be the piecewise linear interpolant at mesh points of $\mathscr{T}^\delta(Q)$. Then by [17], for $0 < \varepsilon < \frac{1}{2}$,

$$\|\Phi^h\|_{\mathrm{H}^{1+\varepsilon}(Q)} \le C\|\Phi\|_{\mathrm{H}^{1+\varepsilon}(Q)}$$

so that by (5.3),

$$\|\Phi^h\|_{\mathrm{H}^{1+\varepsilon}(Q)} \le C\|\varphi\|_{\mathrm{H}^{\frac{1}{2}+\varepsilon}(A)}.$$

The result then follows by the trace theorem since $\Phi^h_{|A} = \varphi^h$. □

Proof of Proposition **5.1**: Let χ be in $\mathrm{H}_0^{\frac{1}{2}+\varepsilon}(A)$, and let χ^h be the linear interpolant of χ at the nodes $(\xi_i)_i$ of t^δ. Then we may write

$$\pi^\delta\chi = \chi^h + \pi^\delta(\chi - \chi^h)$$

Consequently, we have

$$\begin{aligned}\|\pi^\delta\chi\|_{\mathrm{H}^{\frac12}_{00}(A)} &\le \|\chi^h\|_{\mathrm{H}^{\frac12}_{00}(A)} + \|\pi^\delta(\chi-\chi^h)\|_{\mathrm{H}^{\frac12}_{00}(A)}\\ &\le \|\chi^h\|_{\mathrm{H}^{\frac12}_{00}(A)} + \Big(\sum_{i=0}^{i^*-1}\|\pi^\delta(\chi-\chi^h)\|^2_{\mathrm{H}^{\frac12}_{00}(t_i)}\Big)^{\frac12}\\ &\le \|\chi^h\|_{\mathrm{H}^{\frac12}_{00}(A)} + \Big(\sum_{i=0}^{i^*-1}\|\pi^\delta_{t_i}(\chi-\chi^h)\|^2_{\mathrm{H}^{\frac12}_{00}(t_i)}\Big)^{\frac12}.\end{aligned}$$

Using estimate (5.1)

$$\begin{aligned}\|\pi^\delta\chi\|_{\mathrm{H}^{\frac12}_{00}(A)} &\le \|\chi^h\|_{\mathrm{H}^{\frac12}_{00}(A)} + Cp^{\frac14}\Big(\sum_{i=0}^{i^*-1}\|\chi-\chi^h\|^2_{\mathrm{H}^{\frac12+\varepsilon}(t_i)}\Big)^{\frac12}\\ &\le \|\chi^h\|_{\mathrm{H}^{\frac12}_{00}(A)} + Cp^{\frac14}(\|\chi\|_{\mathrm{H}^{\frac12+\varepsilon}(A)} + \|\chi^h\|_{\mathrm{H}^{\frac12+\varepsilon}(A)}).\end{aligned}$$

The proposition follows from the stability (5.2).

Using Proposition 5.1, together with an *hp* extension theorem that allows us to stably lift piecewise polynomials from the boundary into the interior of a meshed domain (Theorem 3.2 of [26]), we then obtain the following estimate. The proof is similar to that of Theorem 3.3 of [26]. For any $\varepsilon > 0$, there exists constants $C_1, C_2 = C_2(\varepsilon)$, independent of u, h and p such that

$$\|u-u^\delta\|_* \le C_2 p^{\frac14} \inf_{v^\delta\in\breve{X}^\delta_u(\Omega)} \sum_{k=1}^{k^*}\|u-v^\delta\|_{\mathrm{H}^{1+\varepsilon}(\Omega_k)} + C_1 \sum_{\Gamma_{k,j}\text{non mortar}} \inf_{\psi^\delta;\psi^\delta\in\mathcal{P}_{p-2}(t^\delta_{k,j})} \left\|\frac{\partial u}{\partial \mathbf{n}} - \psi^\delta\right\|_{\mathrm{H}^{\frac12}_{00}(\Gamma_{k,j}))'}. \tag{5.4}$$

where

$$\breve{X}^\delta_u(\Omega) = \Big\{v^\delta = (v^\delta_k) \in \prod_{k=1}^{k^*} X^\delta(\Omega_k),\quad v^\delta_k(\nu) = u(\nu), \forall\nu \text{ vertex of } \Omega_k\Big\}$$

Note that the first term in (5.4) is simply the consistency error, and the bound in (5.4) for this term is established in the proof of Lemma 4.2.

Theorem 5.4. *Suppose that the hp version of the* (MP) *method is used, with meshes refined geometrically in the vicinity of vertices of the domain* Ω*. Then, the exact solution* u *of (1.2) and the discrete solution* u^δ *of (2.1) satisfy the following estimate*

$$\|u-u^\delta\|_* \le Ce^{-\gamma N^{\frac13}}, \tag{5.5}$$

where N *is the total number of degrees of freedom and* γ *is independant of* N*.*

Proof. It follows from (5.4) and from the best approximation error of the r^α type singularities by *hp* finite elements when geometrical meshes are used (see [5]). □

Remark 5.5. Using (5.4), it may be shown, as was done for (M0) in [26], that the h version with *radical* meshes gives an optimal $O(N^{-\frac{p}{2}})$ asymptotic rate for the approximation error when polynomials of degree p are used, even if the solution has r^α type singularities. However, the consistency error in this case will not be optimal, but will be $O(N^{-\frac{p}{2}+\frac14})$.

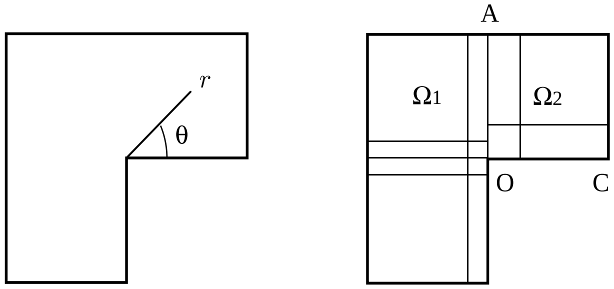

Figure 2. (a) L-Shaped Domains (b) Tensor product mesh for $m = n = 2$.

Remark 5.6. We note that although an optimal convergence estimate can be proved for (MP) in terms of the *hp* version (estimate (4.2)), there is also a small loss of stability of $O(p^{\frac{1}{4}})$, as suggested by Proposition 5.1 (this loss is only in p, with full stability being observed in h). A similar situation occurs for (M0), where we have established the optimal convergence estimate (2.4), but the analog of Proposition 4.1 in [26] shows a stability loss of $O(p^{\frac{3}{4}+\varepsilon})$ (which cannot be avoided, see [27]). The explanation of this apparent contradiction emerges when we consider (M0) and (MP) as mixed methods, with the mortar matching condition taken care by a Lagrange Multiplier. Following the analysis in [8], it is seen that the inf-sup condition of the mixed method behaves essentially like $\|\pi^\delta\|^{-1}_{\mathcal{L}(\mathrm{H}^{\frac{1}{2}}_{00}(A),\mathrm{H}^{\frac{1}{2}}_{00}(A))} = p^{-\frac{1}{4}}$ for (MP) ($p^{-\frac{3}{4}}$ for (M0)). Hence while this loss of stability does not affect the primary unknown u^δ in the mixed method formulation, it may be expected to affect the auxiliary Lagrange multiplier unknown.

6. Numerical Results

We present here the results of the numerical experiments performed by solving the model problem (1.1) on the L-shaped domain shown in Figure 2. We assume that this domain is subdivided in two rectangular subdomains Ω_1 and Ω_2, by the interface AO. For our experiments we impose a Dirichlet boundary condition at the single point C ($\Gamma_D = \{C\}$) and Neumann boundary conditions on $\Gamma_N = \partial\Omega\backslash\Gamma_D$. We employ a mixed method framework (see [8]), and implement the method by relaxing the nodal matching constraints (see Remark 4.3). This is reasonable to do as the approximation error on the primary variable u improves because we have a larger space to choose the infimum. Since the Lagrange multipliers are still approximated by discontinuous piecewise polynomials of degree $p - 2$, the consistency error also does not deteriorate.

We test our method with two different exact solutions, one being smooth and the other unsmooth. For our computations, we consider tensor product meshes where Ω_2 is divided into n^2 rectangular finite elements and Ω_1 is divided into $2m^2$ rectangular finite elements (see Figure 2). In all the graphs, we have plotted (on a log–log scale) the percentage relative broken H^1 error $\|.\|_*$ versus the total number of degrees of freedom of the solution. These plots compare our new method (MP) to the conforming method (CF) as well as the mortar method (M0).

We first choose f so that the exact solution is smooth, and is given by

$$u(r, \theta) = r^4 \cos\left(\frac{2\theta}{3}\right) - 1.$$

We fix the grid with $(m, n) = (4, 6)$ (uniform mesh) and perform a p version by increasing the polynomial degree $3 \le p \le 10$, for both (M0) and (MP). Figure 3 shows that (MP) performs as good as (M0) and the conforming method.

In Figure 4, are plotted the results for which the polynomial degree is fixed at $p = 3$ and different choices of $(m, n) \in \{(2, 3), (4, 6), ..., (12, 18)\}$ are taken, keeping the mesh uniform in both subdomains. We observe

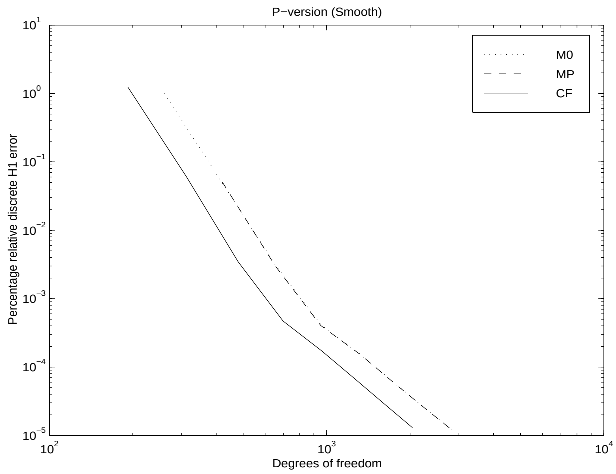

FIGURE 3. p-version for smooth solution.

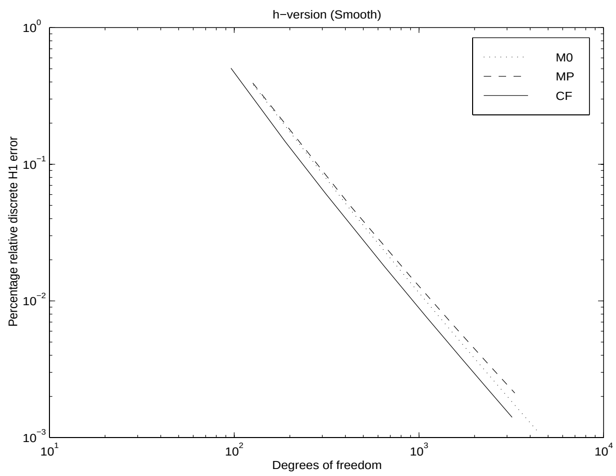

FIGURE 4. h-version for smooth solution, $p = 3$.

that both (M0) and (MP) give optimal $O(h^p)$ rate like the conforming method, though (MP) shows a small deterioration, due to a degraded consistency error.

We now consider our exact solution to be unsmooth, given by,

$$u(r, \theta) = r^{\frac{2}{3}} \cos\left(\frac{2\theta}{3}\right) - 1.$$

It is well-known that the domain in Figure 2 will result in a strong $r^{\frac{2}{3}}$ singularity which occurs at the corner O. The above solution models this. This limits the convergence to $O(h^{\frac{2}{3}})$ when the quasiuniform h version is used. Figure 5 shows that this is the rate observed when degree $p = 3$ elements are used on a uniform mesh.

To improve convergence, we refine the mesh more strongly around O. Let (x_i, y_i) be the coordinates of the mesh points along OC and OA. Then in the *radical mesh* we choose

$$x_i = y_i = \left(\frac{i}{n}\right)^{\beta} \qquad i = 0, 1, \cdots, n,$$

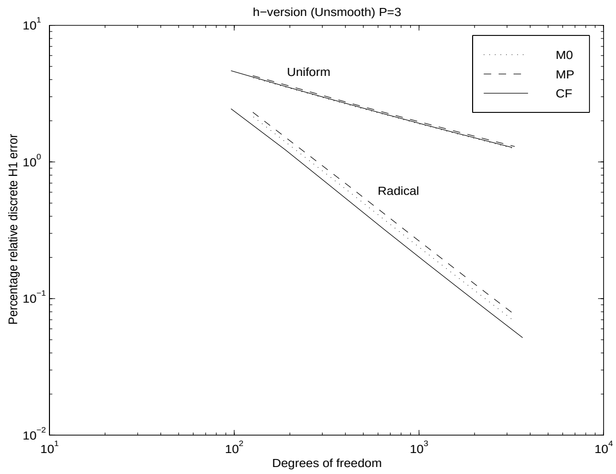

FIGURE 5. h version for unsmooth solution.

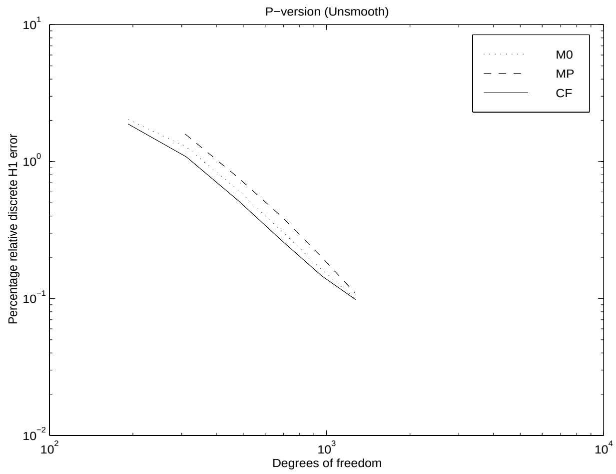

FIGURE 6. p version for unsmooth solution.

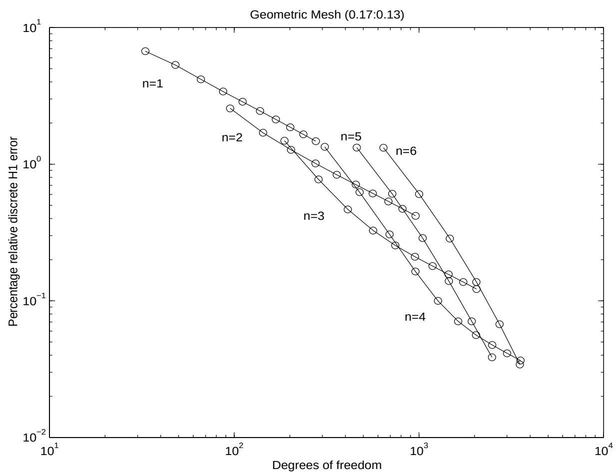

FIGURE 7. hp version for unsmooth solution.

where for $p = 3$, the optimum β is determined experimentally to be 3. (We repeat on Ω_1 with m rather than n). Figure 5 shows that even for such meshes, (MP) (and (M0)) still preserve the improved convergence observed with the conforming method.

Next, we consider *geometric* meshes, where the grid on Ω_2 is designed so that

$$x_0 = y_0 = 0, \quad x_i = y_i = \sigma_2^{n-i}, \quad i = 1, \cdots, n.$$

On the region Ω_1, we take $m = n$, but use a different $\sigma_1 \neq \sigma_2$ to induce non-conformity. The optimal value of σ_1, σ_2 is 0.15 (see [23]), but we take $\sigma_1 = 0.13$ and $\sigma_2 = 0.17$. Figure 6 shows the typical p convergence for increasing p for fixed number of layers n.

Finally, in Figure 7 we perform the hp version for (MP). Here, we plot different p versions, using different number of layers n in the geometric mesh. We see the typical exponential hp behavior if we take the envelope of these curves.

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