DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement 2009

Website: www.aimSciences.org

pp. **424–432**

A STABILITY ESTIMATE FOR FLUID STRUCTURE INTERACTION PROBLEM WITH NON-LINEAR BEAM

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Abstract. In this work we consider the dynamical response of a non-linear beam with viscous damping, perturbed in both the transverse and axial directions interacting with a potential flow. In particular we show that for a class of boundary conditions (clamped beam) and given inlet velocity flow for the fluid, there exists appropriate energy norm for the parameters of the beam (displacements) and flow (potential) bounded by the inlet boundary condition for the fluid flow.

1. **Introduction.** Over many decades, the dynamics of vibrations of non-linear beam structures has been a subject of great interest in the broad field of structural mechanics [15, 11].There have been several classical approaches employed to solve the governing nonlinear differential equations to study the non-linear vibrations including perturbation methods [4], form-function approximations [8], finite element methods [9] and hybrid approaches [10]. In many of these studies, axial deformation was neglected and the average axial force was assumed to be constant over the length of the beam element. However, subsequent analysis showed that axial displacements cannot be neglected in any nonlinear studies [14]. A more comprehensive presentation of the finite element formulation of nonlinear beams can be found in [12, 16].

Of particular interest in recent years, has been the need to develop efficient computational models for the interaction of nonlinear beam structures with fluid media [7, 1, 5]. This is due to the enormous amount of applications of such models in a

2000 *Mathematics Subject Classification.* Primary: 49K20, 70K20; Secondary: 74K10.

Key words and phrases. Stability Analysis, Non-Linear Euler-Bernoulli Beam, Potential Flow, Fluid Structure Interaction.

This work is supported by NSF grant DMS 0813825 and by the ARP grant 0212-44-C399.

variety of areas including biomedical and aerospace applications. Moreover, understanding the fluid-structure interaction between nonlinear beams with fluid will provide better insight into understanding the interaction of other higher dimensional structures with fluids. Although there have been several different computational techniques that are currently being developed and tested for understanding such beam-fluid interactions, there has not yet been a rigorous mathematical stability analysis that has been performed for a nonlinear beam structure interacting with a fluid.

In the current paper we consider a non-linear beam interacting with a two-dimensional potential flow. The beam forms one boundary of the fluid domain and is clamped at the end points. The normal component of the velocity of the fluid on the beam is set equal to the transverse velocity of the beam. In addition we consider transversal forces along the beam according to a pressure function via the Bernoulli equation. These assumptions allow us to study a coupled fluid structure interaction problem as a boundary value problem for a fluid coupled with a non-linear system of equations describing the transverse and axial displacement of the beam. The objective of this paper is to perform a rigorous stability analysis of the associate coupled problem.

The paper is divided into two principle sections. In section 2, we introduce the mathematical model for the dynamic behavior of a nonlinear beam undergoing deformation both in transverse and axial directions. We investigate the stability of the non-linear boundary value problem for axial and transverse displacements of the beam with arbitrary right hand side. The goal is to obtain an energy estimate for the excited non-linear beam in relation to the external forces distributed along the beam. In section 3, we will use the tools from section 2 to get an estimate for the energy of the coupled fluid structure problem under the assumption that the thickness of the beam is constant. The goal of this section is to get an energy estimate for the coupled system in the domain of interest through the integral norm of the velocity of the flow and its potential on the inlet boundary. Note that, we have only considered this model for simplicity and the analysis presented can be extended to other related problems as well. Also, understanding simpler models often provides greater insight to more complex problems.

2. **Mathematical model for the excited non-linear beam.**

In this section we describe the mathematical model of a nonlinear beam of length L clamped at the end points. Let $u(x,t)$ and $w(x,t)$ be the axial and transverse displacements of a generic point on the beam neutral axis (Figure 1). Using the Kirchoff's hypothesis [12], one can express the nonlinear momentum equations as

$$\rho_s A u_{tt} + C_1 u_t - EA\left[u_x + \frac{1}{2}\left(w_x\right)^2\right]_x = 0\,, \tag{1}$$

$$\rho_s A\, w_{tt} + C_1\, w_t - \left[EA\left(u_x + \frac{1}{2}\left(w_x\right)^2\right) w_x\right]_x + E\left(I\, w_{xx}\right)_{xx} = f\,, \tag{2}$$

where ρ_s is the density of the beam, C_1 the damping coefficient, E the Young's modulus, A and I, respectively, the area and the moment of inertia of the beam cross section. In this work, we will assume for simplicity that the beam only experiences a transverse load of f from the fluid and no axial load. Note that this force f corresponds to the pressure exerted by the fluid on the beam (See Figure 1). Dividing Eqs. (1) and (2) by $\rho_s A$ and considering the moment of inertia I to

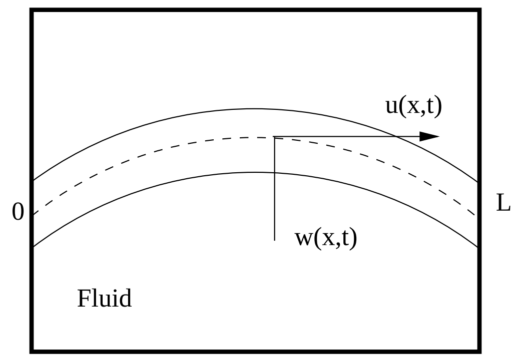

FIGURE 1. Nonlinear beam immersed in a fluid

be constant leads to

$$u_{tt} + K_1\, u_t - D_1 \left[u_x + \frac{1}{2}(w_x)^2 \right]_x = 0\,, \tag{3}$$

$$w_{tt} + K_1\, w_t - D_1 \left[w_x \left(u_x + \frac{1}{2}(w_x)^2 \right) \right]_x + D_2\, w_{xxxx} = q\,, \tag{4}$$

where $K_1 = C_1/\rho_s A$, $D_1 = E/\rho_s$, $D_2 = E\,I/\rho_s A$ and $q = f/\rho_s A$.

2.1. **Energy norm for the nonlinear beam.** In the following we will show that it is possible to build an appropriate energy norm that ensures the boundedness of the beam dynamics with respect to the applied load q.

Multiplying (3) by u_t and by $aK_1 u$, respectively yields,

$$u_{tt}u_t + K_1\, u_t^2 - D_1 \left[u_x + \frac{1}{2}(w_x)^2 \right]_x u_t \quad = \quad 0\,, \tag{5}$$

$$aK_1 u_{tt} u + aK_1^2\, u_t u - aK_1 D_1 \left[u_x + \frac{1}{2}(w_x)^2 \right]_x u \quad = \quad 0\,. \tag{6}$$

where the constant a is chosen between 0 and 1. Multiplying (4) by w_t and $\frac{1}{2}aK_1 w$, respectively yields,

$$w_{tt}w_t + K_1\, w_t^2 - D_1 \left[w_x \left(u_x + \frac{1}{2}(w_x)^2 \right) \right]_x w_t + D_2\, w_{xxxx} w_t \quad = \quad q w_t\,, \tag{7}$$

$$\begin{aligned}\frac{1}{2}aK_1 w_{tt} w + \frac{1}{2}aK_1^2\, w_t w - \frac{1}{2}aK_1 D_1 \left[w_x \left(u_x + \frac{1}{2}(w_x)^2 \right) \right]_x w \quad &+ \\ +\frac{1}{2}aK_1 D_2\, w_{xxxx} w \quad &= \quad \frac{1}{2}aK_1 q w\end{aligned} \tag{8}$$

By adding (5), (6), (7) and (8), integrating in x and by integration by parts, and using the following relations

$$
\begin{aligned}
\frac{1}{2}[u_t^2]_t \quad &+ \quad K_1\, u_t^2 + aK_1 u_{tt} u + \frac{aK_1^2}{2}\left[u^2\right]_t \\
&= \quad \frac{a}{2}\left[(u_t + K_1 u)^2\right]_t + \frac{1-a}{2}[u_t^2]_t + (1-a)K_1\, u_t^2\,, \qquad (9)\\
\frac{1}{2}[w_t^2]_t \quad &+ \quad K_1\, w_t^2 + \frac{aK_1}{2} w_{tt} w + \frac{aK_1^2}{4}\left[w^2\right]_t \\
&= \quad \frac{a}{4}\left[(w_t + K_1 w)^2\right]_t + \left(\frac{1}{2} - \frac{a}{4}\right)[w_t^2]_t + \left(1 - \frac{a}{2}\right) K_1\, w_t^2\,, \qquad (10)
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left[\int_0^L \left(\frac{a}{2}\,(u_t + K_1 u)^2 + \frac{a}{4}\,(w_t + K_1 w)^2 + \frac{1-a}{2} u_t^2 + \left(\frac{1}{2} - \frac{a}{4}\right) w_t^2\right.\right.\\
&\qquad\qquad \left.\left. + \frac{D_1}{2}\left[u_x + \frac{1}{2}(w_x)^2\right]^2 + \frac{D_2}{2} w_{xx}^2\right) dx\right]\\
&+ \int_0^L \left((1-a)K_1\, u_t^2 + \left(1 - \frac{a}{2}\right) K_1\, w_t^2 + aK_1 D_1 \left[u_x + \frac{1}{2}(w_x)^2\right]^2 + \right.\\
&\qquad\qquad \left. + \frac{aK_1 D_2}{2} w_{xx}^2\right) dx = \int_0^L q\left(\frac{aK_1 w}{2} + w_t\right) dx\,. \qquad (11)
\end{aligned}
$$

If we let

$$
\begin{aligned}
V_1(t) \quad := \quad &\int_0^L \left(\frac{a}{2}\,(u_t + K_1 u)^2 + \frac{a}{4}\,(w_t + K_1 w)^2 + \frac{1-a}{2} u_t^2 + \left(\frac{1}{2} - \frac{a}{4}\right) w_t^2\right.\\
&\qquad \left. + \frac{D_1}{2}\left[u_x + \frac{1}{2}(w_x)^2\right]^2 + \frac{D_2}{2} w_{xx}^2\right) dx\,, \qquad (12)
\end{aligned}
$$

and

$$
\begin{aligned}
V_2(t) \quad := \quad &\int_0^L \left((1-a)K_1\, u_t^2 + \left(1 - \frac{a}{2}\right) K_1\, w_t^2 + \right.\\
&\qquad \left. + aK_1 D_1 \left[u_x + \frac{1}{2}(w_x)^2\right]^2 + \frac{aK_1 D_2}{2} w_{xx}^2\right) dx\,, \qquad (13)
\end{aligned}
$$

then from Eq. (11) we will get

$$
(V_1)_t + V_2 = \int_0^L q\left(\frac{aK_1 w}{2} + w_t\right) dx. \qquad (14)
$$

In the particular for the case $a = 0$, equation (14) multiplied back by $\rho_s A$ has a clear physical meaning:

Proposition 1. *For $a = 0$, the rate of change of the sum of the beam kinetic and potential energies $\rho_s A(dV_1/dt)$ plus the dissipated power $\rho_s A(V_2)$ equals the flux of energy given to the beam-system from the fluid flow.*

Next, let us consider the general case $a \neq 0$. Although this case does not have an immediate physical meaning, it allows us to prove an energy estimate which is presented next. By using *Youngs's inequality* [3] we have,

$$q\left(\frac{aK_1 w}{2}+w_t\right) \le \frac{aK_1}{2}\left(\frac{q^2}{4\epsilon_1}+\epsilon_1 w^2\right)+\frac{q^2}{4\epsilon_2}+\epsilon_2 w_t^2\,, \tag{15}$$

and therefore substituting the last inequality in the RHS of Eq. (14) we have

$$\begin{aligned}(V_1)_t \quad &+ \quad \int_0^L \left((1-a)K_1\,u_t^2+\left[\left(1-\frac{a}{2}\right)K_1-\epsilon_2\right]\,w_t^2+\right.\\ &+ \quad \left. aK_1D_1\left(u_x+\frac{1}{2}(w_x)^2\right)^2+\left[\frac{aK_1D_2}{2}w_{xx}^2-\frac{aK_1\epsilon_1}{2}w^2\right]\right)dx\\ &\le \int_0^L q^2\left(\frac{aK_1}{8\epsilon_1}+\frac{1}{4\epsilon_2}\right)dx\,. \end{aligned}\tag{16}$$

From *Poincare inequality* [3]

$$C_F^2\int_0^L w^2\,dx \le \int_0^L w_{xx}^2\,dx\,. \tag{17}$$

For the 1-dimensional domain, C_F can be chosen to be $\frac{1}{L^2}$. If we let

$$\epsilon_1=\frac{D_2}{2L^4}=\frac{C_F^2D_2}{2}, \tag{18}$$

then we obtain

$$\begin{aligned}(V_1)_t \quad &+ \quad \int_0^L \left((1-a)K_1\,u_t^2+\left[\left(1-\frac{a}{2}\right)K_1-\epsilon_2\right]\,w_t^2+\right.\\ &+ \quad \left. aK_1D_1\left(u_x+\frac{1}{2}(w_x)^2\right)^2+\frac{aK_1D_2}{4}w_{xx}^2\right)dx\\ &\le \quad \int_0^L q^2\left(\frac{aK_1}{8\epsilon_1}+\frac{1}{4\epsilon_2}\right)dx\,. \end{aligned}\tag{19}$$

Finally by choosing

$$\epsilon_2=\frac{\left(1-\frac{a}{2}\right)K_1}{2}\,, \tag{20}$$

and letting

$$\begin{aligned}V_2^*(t) \quad := \quad &\int_0^L \left((1-a)\,u_t^2+\frac{1}{2}\left(1-\frac{a}{2}\right)\,w_t^2+\right.\\ &+\ \left. aD_1\left(u_x+\frac{1}{2}(w_x)^2\right)^2+\frac{aD_2}{4}w_{xx}^2\right)dx, \end{aligned}\tag{21}$$

the following inequality follows

$$(V_1)_t+K_1V_2^* \le \int_0^L q^2\left(\frac{aK_1}{8\epsilon_1}+\frac{1}{4\epsilon_2}\right)dx\,. \tag{22}$$

Integrating in time from 0 to t leads to

$$V_1(t)-V_1(0)+K_1\int_0^t V_2^*(\tau)\,d\tau \le \left(\frac{aK_1}{8\epsilon_1}+\frac{1}{4\epsilon_2}\right)\int_0^t \|q\|_{L_2}^2 d\tau\,, \tag{23}$$

where the L_2 norm of q is taken in $[0, L]$. Observing that to each square term in the V_2^* (21) there corresponds an analogous term in the V_1 (12), it is not difficult to see that

$$V_2^* \leq 2V_1. \tag{24}$$

and combining (23) and (24) leads to the following inequality

$$\int_0^T V_2^*(t)\, dt \leq \frac{V_1(0)}{K_1}[1 - e^{-2K_1T}] + \\ +2\left(\frac{aK_1}{8\epsilon_1} + \frac{1}{4\epsilon_2}\right) e^{-2K_1T} \int_0^T e^{2K_1t} \int_0^t \|q\|_{L_2}^2 d\tau\, dt \tag{25}$$

Therefore we have the following result:

Theorem 2.1. *Let the nonlinear beam be excited by a distributed transversal load q, then the energy V_2^* satisfies inequality (25).*

From Eq. (25) the following stability results can be obtained:

Remark 1. Assume $\|q\|_{L_2}$ to be bounded by C for all time, then Inequality (25) leads to

$$\int_0^T V_2^*(t)\, dt \leq \frac{V_1(0)}{K_1}[1 - e^{-2K_1T}] + \\ +\left(\frac{aK_1}{8\epsilon_1} + \frac{1}{4\epsilon_2}\right) \frac{C}{K_1}\left[T + e^{-2K_1T} - 1\right] \leq C_0 + C_1 T\,. \tag{26}$$

In addition, if $\lim_{t\to\infty} V_2^*(t) = A$, then $A < \infty$.

Remark 2. Assume $\int_0^\infty \|q\|_{L_2} dt = C$, then Inequality (25) leads to

$$\int_0^T V_2^*(t)\, dt \leq \frac{V_1(0)}{K_1}[1 - e^{-2K_1T}] + \\ +\left(\frac{aK_1}{8\epsilon_1} + \frac{1}{4\epsilon_2}\right) \frac{C}{K_1}\left[1 - e^{-2K_1T}\right] \leq C_0\,. \tag{27}$$

In addition, if $\lim_{t\to\infty} V_2^*(t) = A$, then $A = 0$.

3. **Beam-fluid interaction.** We will now formulate the fluid-beam interaction problem, where the beam is modeled as the top boundary of the 2-dimensional fluid flow domain Ω. Let

- Ω be the fluid domain: $\Omega = \{(x, y), 0 < x < L, -h < y < 0\}$,
- Γ be the beam (top): $\Gamma = \{(x, y), 0 < x < L, y = 0\}$,
- Γ_1 be the inlet region (left): $\Gamma_1 = \{(x, y), x = 0, -h < y < 0\}$,
- Γ_2 be the outlet region (right): $\Gamma_2 = \{(x, y), x = L, -h < y < 0\}$,
- Γ_3 be the impermeable boundary (bottom): $\Gamma_3 = \{(x, y), 0 < x < L, y = -h\}$,
- H be the total extent of the domain in the third direction (orthogonal to x and y).

We consider an inviscid, incompressible and irrotational flow with velocity $\vec{v} = (v_1, v_2)$ equal to the gradient of the potential flow $\phi(x, y, t)$ and satisfying the Laplace equation

$$\Delta\phi = 0 \qquad \text{on } \Omega\,, \tag{28}$$

with boundary conditions

$$\phi_x = b(y,t) \quad \text{on } \Gamma_1\,, \tag{29}$$
$$\phi_x = -\alpha\phi_t \quad \text{on } \Gamma_2\,, \tag{30}$$
$$\phi_y = 0 \quad \text{on } \Gamma_3\,. \tag{31}$$

On the top boundary Γ the coupled system satisfies the continuity constraints

$$u_t = \phi_x = v_1, \tag{32}$$
$$w_t = \phi_y = v_2, \tag{33}$$

and the force balance equations

$$\rho_s A u_{tt} + C_1 u_t - EA\left[u_x + \frac{1}{2}w_x^2\right]_x = 0\,, \tag{34}$$
$$\rho_s A w_{tt} + C_1 w_t - EA\left[\left(u_x + \frac{1}{2}w_x^2\right)w_x\right]_x + EI w_{xxxx} = p\,H\,, \tag{35}$$

where p is pressure exerted from the fluid to the beam. According to the Bernoulli equation [2, 6] the pressure p satisfies

$$p = -\rho_f\phi_t - \rho_f g\,w\,, \tag{36}$$

where ρ_f is the density of the fluid and g is the gravity acceleration.

Our goal is to obtain an energy estimate for the beam-fluid coupled system in the region $\Omega\cup\Gamma$ through the data available in the inlet boundary Γ_1. This is described next. Multiplying (28) by ϕ_t and applying the *Green's identity* [3]we get,

$$\begin{aligned} 0 &= \int_\Omega \Delta\phi\,\phi_t\,d\mathbf{x} \\ &= -\frac{1}{2}\int_\Omega \left[(\nabla\phi)^2\right]_t\,d\mathbf{x} + \int_\Gamma \phi_y\phi_t\,dx - \int_{\Gamma_1}\phi_x\phi_t\,dy + \int_{\Gamma_2}\phi_x\phi_t\,dy \\ &= -\frac{1}{2}\int_\Omega \left[(\nabla\phi)^2\right]_t\,d\mathbf{x} + + \int_\Gamma \phi_y\phi_t\,dx - \int_{\Gamma_1} b\,\phi_t\,dy - \frac{1}{\alpha}\int_{\Gamma_2}\phi_x^2\,dy. \end{aligned} \tag{37}$$

Multiplying by $H\rho_f$ and rearranging the terms we obtain,

$$\begin{aligned} -H\rho_f\int_\Gamma \phi_y\phi_t\,dx = -\frac{H\rho_f}{2}\int_\Omega\left[(\nabla\phi)^2\right]_t d\mathbf{x} \quad &- \quad H\rho_f\int_{\Gamma_1} b\phi_t\,dy \\ &- \quad \frac{H\rho_f}{\alpha}\int_{\Gamma_2}\phi_x^2\,dy\,. \end{aligned} \tag{38}$$

For the non-linear beam on the boundary Γ one can build an equation analogous to the equation (11) with $a = 0$ and distributed load equal to $f = H\,p$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Gamma\left(\rho_s A(u_t^2 + w_t^2) + EA\left(u_x + \frac{1}{2}(w_x)^2\right)^2 + EI\,w_{xx}^2\right)dx \\ &\qquad + \int_\Gamma C_1(u_t^2 + w_t^2)\,dx = \int_\Gamma H\,pw_t\,dx \\ &\qquad\qquad = -H\rho_f\int_\Gamma gww_t\,dx - H\rho_f\int_\Gamma \phi_t\phi_y\,dx\,. \end{aligned} \tag{39}$$

Observing that the second term in the right hand side of (39) is exactly the left hand side of (38) and rearranging all the terms, it follows

$$\frac{1}{2}\frac{d}{dt}\left[\int_\Gamma\left(\rho_s A(u_t^2+w_t^2)+EA\left(u_x+\frac{1}{2}(w_x)^2\right)^2+EI\,w_{xx}^2+H\rho_f g w^2\right)dx\right.$$
$$\left.+\int_\Omega H\rho_f(\nabla\phi)^2\,d\mathbf{x}\right]+\int_\Gamma C_1(u_t^2+w_t^2)\,dx+\int_{\Gamma_2}\frac{H\rho_f}{\alpha}\phi_x^2\,dy$$
$$=-\int_{\Gamma_1}H\rho_f b\,\phi_t\,dy. \tag{40}$$

If we define the following two energy norms

$$I_1(t):=\frac{1}{2}\left[\int_\Gamma\left(\rho_s A(u_t^2+w_t^2)+EA\left(u_x+\frac{1}{2}(w_x)^2\right)^2+EI\,w_{xx}^2+H\rho_f g w^2\right)dx+\right.$$
$$\left.+\int_\Omega H\rho_f(\nabla\phi)^2\,d\mathbf{x}\right], \tag{41}$$

and

$$I_2(t):=\int_\Gamma C_1(u_t^2+w_t^2)\,dx+\int_{\Gamma_2}\frac{H\rho_f}{\alpha}\phi_x^2\,dy\,, \tag{42}$$

then we have,

$$\frac{dI_1}{dt}+I_2=-\int_{\Gamma_1}H\rho_f b\,\phi_t\,dy. \tag{43}$$

Integrating Eq. (43) in time from 0 to T gives the cumulative energy form

$$I_1(T)-I_1(0)+\int_0^T I_2(t)\,dt=-\int_0^T\int_{\Gamma_1}H\rho_f b\,\phi_t\,dy\,. \tag{44}$$

Eq. (40), or its integral form (44), has a clear physical meaning:

Theorem 3.1. *The rate of change of the sum of all kinetic and potential energies (dI_1/dt) plus the sum of the dissipated power and the flux of outgoing energy (I_2) equals the flux of incoming energy (RHS).*

4. **Conclusion.**

In this paper, the dynamic behavior of a non-linear beam coupled with a potential flow has been studied. Suitable energy norms for analyzing the dynamics of the non-linear beam by itself as well as for the fluid-beam coupled problem were developed and corresponding stability estimates were proved. The estimate helps us to effectively quantify and evaluate the cumulative energy of the coupled system in the whole region of interest. Although, these estimates were presented in this paper for a beam-fluid interaction, one can extend the methods presented to more complex problems involving dynamic interactions. The latter will be investigated in our future work.

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Received July 2008; revised March 2009.

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