COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **10**, Number **5**, September **2011**

doi:10.3934/cpaa.2011.10.1447

pp. **1447–1462**

STABILITY ANALYSIS OF INHOMOGENEOUS EQUILIBRIUM FOR AXIALLY AND TRANSVERSELY EXCITED NONLINEAR BEAM

Emine Kaya-Cekin, Eugenio Aulisa and Akif Ibragimov

Department of Mathematics and Statistics, Texas Tech University,
Lubbock TX, 79409-1042

Padmanabhan Seshaiyer

Department of Mathematical Sciences, George Mason University,
Fairfax VA, 22030, USA

Abstract. In this work we consider the dynamical response of a non-linear beam with viscous damping, perturbed in both the transverse and axial directions. The system is modeled using coupled non-linear momentum equations for the axial and transverse displacements. In particular we show that for a class of boundary conditions (beam clamped at the extremes) and uniformly distributed load, there exists a non-uniform equilibrium state. Different models of damping are considered: first, third and fifth order dissipation terms. We show that in all cases in the presence of the damping forces, the excited beam is stable near the equilibrium for any perturbation. An energy estimate approach is used in order to identify the space in which the solution of the perturbed system is stable.

1. **Introduction.** Over many decades, the dynamics of vibrations of nonlinear beam structures has been a subject of great interest in the broad field of structural mechanics [28, 22]. There have been several classical approaches employed to solve the governing nonlinear differential equations to study the nonlinear vibrations including perturbation methods [9], form-function approximations [18], finite element methods [19, 21] and hybrid approaches [20]. In many of these studies, axial deformation was neglected and the average axial force was assumed to be constant over the length of the beam element. However, subsequent analysis showed that axial displacements cannot be neglected in any nonlinear studies [27]. A more comprehensive presentation of the finite element formulation of nonlinear beams can be found in [23, 29].

Of particular interest in recent years, there has been the need to develop efficient computational models for the interaction of nonlinear beam structures with fluid media [1, 2, 3, 4, 5, 6, 11, 16]. This is due to the large amount of applications of such models in a variety of areas including biomedical ([13, 25, 14] and references herein) and aerospace applications [10, 26]. Moreover, understanding the fluid-structure interaction between nonlinear beams with fluid will provide better insight into understanding the interaction of other higher dimensional structures

2000 *Mathematics Subject Classification.* Primary: 49K20, 70K20; Secondary: 74K10.

Key words and phrases. Stability analysis, non-linear partial differential equations, Euler-Bernoulli beam.

This work is supported by NSF grant DMS 0813825.

with fluids. Although there have been several different computational techniques that are currently being developed and tested for understanding such beam-fluid interactions, there has not yet been a rigorous mathematical stability analysis that has been performed for a nonlinear beam structure interacting with a fluid. We believe that to solve the latter, it is essential to understand and perform a rigorous stability analysis of the associate structural mechanics problem, namely, the dynamic behavior of a nonlinear beam undergoing deformation both in transverse and axial directions, which will be the focus of the paper. In the original paper of Dickey [7], the stability of a non-linear evolutionary beam equation is studied under the assumption that the axial displacement is negligible and the original system is reduced to a non-linear equation with damping for the vertical displacement only. The most comprehensive stability analysis for a non-linear plate can be found in [17]. There the contribution of the in-plane velocities are neglected which allow to reduce the original system to a simplified system for the vertical displacement and the Airy stress. The dissipative terms are then introduced into system by way of nonlinear state feedback equation on the edge of the plate.

Toward this end, we consider a general theoretical model of a nonlinear beam with viscous damping that deforms both in the axial and transverse directions due to an external applied force, possibly from a fluid. The mathematical model considered herein involves a coupled system of partial differential equations describing the axial and transverse displacements of a nonlinear Euler-Bernoulli beam with viscous damping. This coupled system is developed in Section 2 and a stationary solution using a fixed point approach is presented in Section 3. Next, in Section 4, the coupled system of equations is linearized about the stationary obtained solution. Section 5 presents the key result on stability for the system obtained. In particular, we show that for the coupled system, for a class of boundary conditions there exists a non-uniform equilibrium state of the beam excited with uniformly distributed force. It is rigorously proved using an energy estimates approach, that in the presence of the damping forces, the excited beam is stable near equilibrium for any perturbation. In addition it was observed that if the damping coefficient is vanishing in time faster than 1/time, then system may resonate even with bounded right hand side.

2. **Model problem and background.**

Consider a mathematical model for a geometrically nonlinear beam of length L clamped at the end points. Let $u(x,t)$ and $w(x,t)$ be the respective axial and transverse displacements of a point on the neutral axis of the nonlinear beam (Figure 1). Note that, we have only considered this model for simplicity and the analysis presented can be extended to other related problems as well. Also, understanding simpler models often provides greater insight to further investigate more complex problems.

Using Kirchoff's hypothesis [23], one can express the displacement at any given point (u_1, u_2, u_3) in the beam in terms of the axial and transverse deflections along the neutral axis as:

$$u_1(x,y,t) = u(x,t) - y\frac{\partial w}{\partial x}\,, \tag{1}$$

$$u_2(x,t) = w(x,t)\,, \tag{2}$$

$$u_3(x,t) = 0\,. \tag{3}$$

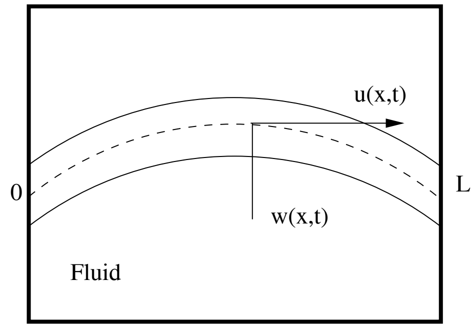

FIGURE 1. Nonlinear beam immersed in a fluid

Using the nonlinear strain displacement relations [12]

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_m}{\partial x_i}\,\frac{\partial u_m}{\partial x_j}\right), \tag{4}$$

and omitting the large strain terms but only retaining the square of $\partial u_2/\partial x$ (which represents the rotation of a transverse normal line in the beam) one can easily verify that

$$\epsilon_{xx} = \frac{\partial u}{\partial x} - y\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \tag{5}$$

$$\epsilon_{yy} = \epsilon_{xy} = 0\,. \tag{6}$$

For the constitutive relationship, we now employ the linearized material law that relates the stress to the non-linear strain given by $\sigma_{xx} = E\,\epsilon_{xx}$ (note that one can also consider nonlinear material laws, which will be investigated in forthcoming papers and will not be considered in this paper).

Using the principle of virtual work [23], one can then derive the following governing equations describing the motion of the nonlinear beam structure to be

$$\rho A u_{tt} + C_1 u_t - EA\left[u_x + \frac{1}{2}(w_x)^2\right]_x = 0\,, \tag{7}$$

$$\begin{aligned}\rho A\, w_{tt} + C_1\, w_t - C_2 w_{txx} + C_3\, w_{txxxx}&\\ - \left[EA\left(u_x + \frac{1}{2}(w_x)^2\right) w_x\right]_x + EI_0\, w_{xxxx} &= f\,,\end{aligned} \tag{8}$$

where ρ is the density of the beam, C_1, C_2 and C_3 the damping coefficients, E the Young's modulus, A and I_0, respectively, the area and the momentum of inertia of the beam cross section. In Eq. (8) the higher orders damping terms $C_2 w_{txx}$, and $C_3 w_{txxxx}$ have been introduced according to [24], where the it was stated that "the simple (first-order) viscous damping models ... are inadequate if experimentally observed damping properties are to be incorporated in the model". These higher order terms were first introduced by Lord Kelvin and Robert Voigt. In their hypothesis the internal dumping forces may be described with a positive multiple of the elastic resonance forces acting on velocity rather than displacement.

Depending of the values of the constants C_1, C_2 and C_3 each of the models reported in [24] can be reproduced. In this work, we will assume for simplicity that the beam only experiences a transverse load of f and no axial load. Note that this force f may correspond to the pressure exerted by the fluid on the beam (See Figure 1). However for the analysis of the associated fluid-structure interaction problem, one must consider (7,8) coupled with fluid equations (see for example [3, 15]).

Dividing system (7-8) by ρA reduces to

$$u_{tt} + K_1\, u_t - D_1\left[u_x + \tfrac{1}{2}(w_x)^2\right]_x = 0\,, \tag{9}$$

$$w_{tt}+K_1 w_t-K_2 w_{txx}+K_3 w_{txxxx}-D_1\left[w_x\left(u_x+\tfrac{1}{2}(w_x)^2\right)\right]_x+D_2 w_{xxxx}=q, \tag{10}$$

where $K_1 = C_1/\rho A$, $K_2 = C_2/\rho A$, $K_3 = C_3/\rho A$, $D_1 = E/\rho$, $D_2 = E\,I_0/\rho A$ and $q = f/\rho A$. In this paper, we will present a detailed stability analysis of the coupled system (9-10) Next we consider the stationary solution to this coupled system.

3. **Stationary solution of the non-linear system.**

Consider the steady state system of equations (9-10) on $[0, L]$ given by

$$-D_1\left[U_x + \frac{1}{2}(W_x)^2\right]_x = 0 \tag{11}$$

$$-D_1\left[W_x\left(U_x + \frac{1}{2}(W_x)^2\right)\right]_x + D_2 W_{xxxx} = q \tag{12}$$

where $W = W(x)$ and $U = U(x)$ are only functions of x (therefore all the time derivatives are identically zero). Consider a uniformly distributed load q. Since the beam is assumed to be clamped at the end points, we have the following set of boundary conditions

$$U(0) = U(L) = 0, \tag{13}$$

$$W(0) = W(L) = W'(0) = W'(L) = 0. \tag{14}$$

Then (11-12) yields

$$U_x + \tfrac{1}{2}(W_x)^2 = C, \tag{15}$$

$$-D_1\,(C W_{xx}) + D_2 W_{xxxx} = q, \tag{16}$$

for a unique constant C. It is not difficult to see, by using Rolle's theorem and the boundary conditions of U, that there exists at least one point x_0 where $U_x(x_0) = 0$. This ensures the positivity of the constant C. By solving Eq. (16) using boundary conditions (14), we obtain

$$W(x) = \frac{q\left((C\,D_1)^{1/2}(L-x)x+D_2^{1/2}L\big(\cosh(\alpha(L-2x)/2)\operatorname{csch}(\alpha L/2)-\coth(\alpha L/2)\big)\right)}{2(CD_1)^{3/2}}, \tag{17}$$

where $\alpha = (CD_1/D_2)^{1/2}$. Substituting this last expression for W_x^2 in (11) and using the boundary conditions (13) allow to solve for U

$$\begin{aligned} U(x) = {} & \frac{q^2}{48(C\,D_1)^3\,(\cosh(\alpha L)-1)}\big\{4(L-2x)(6D_2 + CD_1(L-x)x)(1-\cosh(\alpha L)) \\ & +3D_2^{1/2}L\big[8D_2^{1/2}(\cosh(\alpha(L-x)) - \cosh(\alpha x)) + +(CD_1)^{1/2}\big(3(L-2x)\sinh(\alpha L) \\ & +L\sinh(\alpha(L-2x)) - 4(L-2x)(\sinh(\alpha(L-x)) + \sinh(\alpha x))\big)\big]\big\}\,. \end{aligned} \tag{18}$$

The constant C should be chosen to satisfy Eq. (15) which can be rewritten as a fixed point equation

$$U_x + \frac{1}{2}(W_x)^2 = F(C) = C\,, \tag{19}$$

where

$$F(C) = \frac{q^2\left(48D_2 + 2CD_1L^2 - 18\sqrt{CD_1}\sqrt{D_2}\coth(\alpha L/2) - 3CD_1L^2\mathrm{csch}(\alpha L/2)^2\right)}{48C^3D_1^3L}\,. \tag{20}$$

Note that $F(C)$ is a continuous functions for all $C > 0$. Moreover $\lim_{C\to 0} F(C) = \frac{L^6q^2}{60480D_2^2} > 0$ and $\lim_{C\to\infty} F(C) = 0$. As a consequence there exists at least one C_0 $(0 < C_0 < \infty)$ for which $F(C_0) = C_0$. This fact guarantees existence of the steady-state solution. A further analysis on the sign of $F'(C)$ suggests monotonicity of the function $F(C)$ and indeed uniqueness of the solution. In Figure 2 we show the graphs of the steady state solution (U,W) for $D_1 = 1$, $D_2 = 1$, $Q = 1$ and $L = 1$. For this particular case the value of C is 1.6534×10^{-5}.

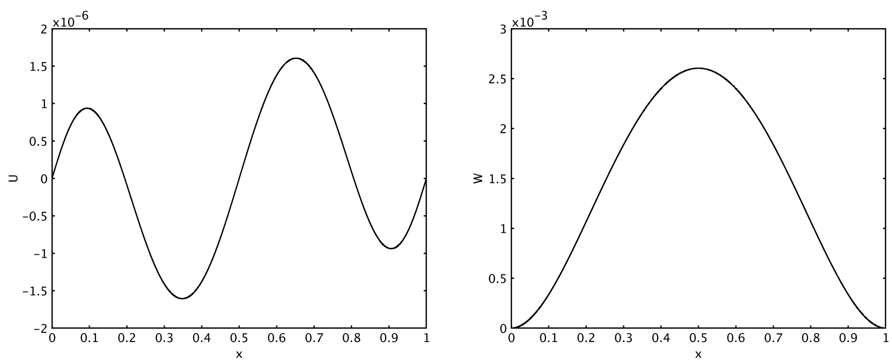

FIGURE 2. Nonlinear beam steady state solution, for $D_1 = 1$, $D_2 = 1$, $Q = 1$ and $L = 1$.

In the next section, we consider the linearization of the system (7-8) about the stationary solution.

4. **Linearization about the stationary solution.** Consider the Taylor expansions of the functions $(w_x)^2$ and $w_x\left[u_x + \frac{1}{2}(w_x)^2\right]$ around the stationary solutions W_x and (W_x, U_x) respectively. We then have,

$$(w_x)^2 = (W_x)^2 + 2W_x(w_x - W_x) + O(w_x - W_x)^2\,, \tag{21}$$

and

$$\begin{aligned} w_x\left[u_x + \frac{1}{2}(w_x)^2\right] &= W_x\left[U_x + \frac{1}{2}(W_x)^2\right] + W_x(u_x - U_x) \\ &+ \left[U_x + \frac{1}{2}(W_x)^2 + (W_x)^2\right](w_x - W_x) \\ &+ O\left[(w_x - W_x)^2, (u_x - U_x)^2, (w_x - W_x)(u_x - U_x)\right]\,, \end{aligned} \tag{22}$$

which reduces using (15) to

$$\begin{aligned} w_x\left[u_x + \frac{1}{2}(w_x)^2\right] &= CW_x + W_x(u_x - U_x) + [C + (W_x)^2](w_x - W_x) \\ &+ O\left[(w_x - W_x)^2, (u_x - U_x)^2, (w_x - W_x)(u_x - U_x)\right]\,. \end{aligned} \tag{23}$$

Let $\epsilon = u - U$ and $\delta = w - W$, then $\epsilon_x = u_x - U_x$ and $\delta_x = w_x - W_x$. Assuming ϵ, δ, ϵ_x and δ_x small enough, we can neglect the second order terms and rewrite equations (21) and (23) as

$$(w_x)^2 = (W_x)^2 + 2W_x\delta_x \,, \tag{24}$$

$$w_x\left[u_x + \frac{1}{2}(w_x)^2\right] = CW_x + W_x\epsilon_x + [C + (W_x)^2]\delta_x \,. \tag{25}$$

Substituting Eq. (24) into Eq. (9) yields

$$\epsilon_{tt} + K_1\epsilon_t - D_1\left[\epsilon_x + U_x + \frac{1}{2}(W_x)^2 + W_x\delta_x\right]_x = 0\,, \tag{26}$$

and since $\left[U_x + \frac{1}{2}(W_x)^2\right]_x = 0$, we have

$$\epsilon_{tt} + K_1\epsilon_t - D_1\left[\epsilon_x + W_x\delta_x\right]_x = 0\,. \tag{27}$$

Similarly substituting (25) into (10) we get

$$\begin{aligned}&\delta_{tt} + K_1\delta_t - K_2\delta_{txx} + K_3\delta_{txxxx}\\ &\quad - D_1\left[CW_x + W_x\epsilon_x + [C + (W_x)^2]\delta_x\right]_x + D_2[\delta_{xxxx} + W_{xxxx}] = q\,.\end{aligned} \tag{28}$$

Substituting Eq. (16) into Eq. (28) yields

$$\delta_{tt} + K_1\delta_t - K_2\delta_{txx} + K_3\delta_{txxxx} - D_1\left[W_x\epsilon_x + [C + (W_x)^2]\delta_x\right]_x + D_2\delta_{xxxx} = 0\,. \tag{29}$$

Now by letting $a(x) = U_x$, $b(x) = W_x$ and $d(x) = C + (W_x)^2$, we obtain the following linearized system

$$\epsilon_{tt} + K_1\epsilon_t - D_1\epsilon_{xx} - D_1[b(x)\delta_x]_x = 0\,, \tag{30}$$

$$\delta_{tt} + K_1\delta_t - K_2\delta_{txx} + K_3\delta_{txxxx} - D_1[b(x)\epsilon_x]_x - D_1[d(x)\delta_x]_x + D_2\delta_{xxxx} = 0. \tag{31}$$

5. A stability result.

In this section, we prove our main result for the solution of the linearized system above.

Definition 5.1. (**Monotonic stability**) We call an equilibrium state monotonically stable [8], if there exists a Lyapunov function $I(\epsilon, \delta, t)$ such that

$$I(\epsilon, \delta, t_1) \geq I(\epsilon, \delta, t_2) \qquad \text{if} \qquad t_1 < t_2\,. \tag{32}$$

Definition 5.2. (**Asymptotic stability**) We call an equilibrium state asymptotically stable [8], if there exists a Lyapunov function $I(\epsilon, \delta, t) \geq 0$ such that

$$I(\epsilon, \delta, t) = 0 \qquad \text{iff} \qquad \epsilon = \delta = 0\,, \tag{33}$$

$$I(\epsilon, \delta, t) \to 0 \qquad \text{for} \qquad t \to \infty\,. \tag{34}$$

We will prove that our *linearized* system is both monotonically and asymptotically stable for any initial perturbation $(\epsilon, \delta, 0)$. To accomplish this, we will first manipulate the linearized system (30-31) to develop a suitable energy norm that will be used in our stability analysis.

Multiplying (30) by ϵ_t and $K_1\epsilon$ we have respectively,

$$\tfrac{1}{2}(\epsilon_t^2)_t + K_1\epsilon_t^2 - D_1\epsilon_t\epsilon_{xx} - D_1\epsilon_t\left[b(x)\delta_x\right]_x = 0\,, \tag{35}$$

$$K_1\epsilon_{tt}\epsilon + \tfrac{1}{2}K_1^2(\epsilon^2)_t - K_1D_1\epsilon\epsilon_{xx} - K_1D_1\epsilon\left[b(x)\delta_x\right]_x = 0\,. \tag{36}$$

Let $B_1 = \frac{1}{2}(\epsilon_t^2)_t + K_1\epsilon_t^2$ and $B_2 = K_1\epsilon_{tt}\epsilon + \frac{1}{2}K_1^2(\epsilon^2)_t$, then

$$\begin{aligned}B_1 + B_2 &= \frac{1}{2}(\epsilon_t^2)_t + K_1\epsilon_{tt}\epsilon + K_1\epsilon_t^2 + \frac{1}{2}K_1^2(\epsilon^2)_t\\ &= \frac{1}{2}(\epsilon_t^2)_t + K_1\left(\epsilon_t\epsilon\right)_t + \frac{1}{2}K_1^2(\epsilon^2)_t\\ &= \left[\frac{1}{2}(\epsilon_t + K_1\epsilon)^2\right]_t = [B(\epsilon)]_t\,.\end{aligned} \tag{37}$$

Adding (35) and (36) and integrating with respect to x, we have,

$$\begin{aligned}\left[\int_0^L B(\epsilon)\,dx\right]_t &= D_1\int_0^L \epsilon_t\epsilon_{xx}\,dx + D_1\int_0^L \epsilon_t\left[b(x)\delta_x\right]_x\,dx\\ &\quad +K_1D_1\int_0^L \epsilon\epsilon_{xx}\,dx + K_1D_1\int_0^L \epsilon\left[b(x)\delta_x\right]_x\,dx\,. \qquad (38)\end{aligned}$$

Using integration by parts for the terms on the right hand side along with the boundary conditions $\epsilon(0)=\epsilon(L)=\epsilon_t(0)=\epsilon_t(L)=0$, reduces (38) to

$$\begin{aligned}&\left[\int_0^L B(\epsilon)\,dx\right]_t + \frac{D_1}{2}\left[\int_0^L \epsilon_x^2\,dx\right]_t\\ &= -D_1\int_0^L b(x)\delta_x\epsilon_{xt}\,dx - K_1D_1\int_0^L \epsilon_x^2\,dx - K_1D_1\int_0^L b(x)\delta_x\epsilon_x\,dx\,. \qquad (39)\end{aligned}$$

Next, multiplying (31) by δ_t and $K_1\delta$ respectively, we get,

$$\begin{aligned}&\frac{1}{2}(\delta_t^2)_t + K_1(\delta_t)^2 - K_2\delta_{txx}\delta_t + K_3\delta_{txxxx}\delta_t\\ &\quad -D_1\left[b(x)\epsilon_x\right]_x\delta_t - D_1\left[d(x)\delta_x\right]_x\delta_t + D_2(\delta_{xx})_{xx}\delta_t = 0\,, \qquad (40)\end{aligned}$$

and

$$\begin{aligned}&K_1\delta_{tt}\delta + \frac{K_1^2}{2}(\delta^2)_t - K_1K_2\delta_{txx}\delta + K_1K_3\delta_{txxxx}\delta -\\ &\quad -K_1D_1\left[b(x)\epsilon_x\right]_x\delta - K_1D_1\left[d(x)\delta_x\right]_x\delta + K_1D_2(\delta_{xx})_{xx}\delta = 0\,. \qquad (41)\end{aligned}$$

Let $B_3 = \frac{1}{2}(\delta_t^2)_t + K_1(\delta_t)^2$ and $B_4 = K_1\delta_{tt}\delta + \frac{K_1^2}{2}(\delta^2)_t$, then

$$\begin{aligned}B_3 + B_4 &= \frac{1}{2}(\delta_t^2)_t + K_1(\delta_t)^2 + K_1\delta_{tt}\delta + \frac{K_1^2}{2}(\delta^2)_t\\ &= \frac{1}{2}(\delta_t^2)_t + K_1(\delta_t\delta)_t + \frac{K_1}{2}(\delta^2)_t\\ &= \left[\frac{1}{2}(\delta_t + K_1\delta)^2\right]_t = \left[B(\delta)\right]_t\,. \qquad (42)\end{aligned}$$

Adding (40) and (41) and integrating with respect to x, we have,

$$\begin{aligned}&\left[\int_0^L B(\delta)\,dx\right]_t = D_1\int_0^L \left[b(x)\epsilon_x\right]_x\delta_t\,dx + D_1\int_0^L \left[d(x)\delta_x\right]_x\delta_t\,dx\\ &-D_2\int_0^L (\delta_{xx})_{xx}\delta_t\,dx + K_1D_1\int_0^L \left[b(x)\epsilon_x\right]_x\delta\,dx + K_1D_1\int_0^L \left[d(x)\delta_x\right]_x\delta\,dx\\ &-K_1D_2\int_0^L (\delta_{xx})_{xx}\delta\,dx + K_2\int_0^L \delta_{txx}\delta_t\,dx + K_1K_2\int_0^L \delta_{txx}\delta\,dx\\ &-K_3\int_0^L \delta_{txxxx}\delta_t\,dx - K_1K_3\int_0^L \delta_{txxxx}\delta\,dx\,. \qquad (43)\end{aligned}$$

Then applying integration by parts to the terms in the right hand side and using the boundary conditions $\delta(0)=\delta(L)=\delta_x(0)=\delta_x(L)=0$ ($\Longrightarrow \delta_{xt}(0)=\delta_{xt}(L)=0$), we obtain

$$\begin{aligned}&\left[\int_0^L B(\delta)\,dx\right]_t + \frac{K_1K_2}{2}\left[\int_0^L \delta_x^2\,dx\right]_t + \frac{K_1K_3}{2}\left[\int_0^L \delta_{xx}^2\,dx\right]_t\\ &+\frac{D_2}{2}\left[\int_0^L \delta_{xx}^2\,dx\right]_t = -K_2\int_0^L \delta_{tx}^2\,dx - K_3\int_0^L \delta_{txx}^2\,dx\\ &-K_1D_2\int_0^L \delta_{xx}^2\,dx - D_1\int_0^L b(x)\epsilon_x\delta_{xt}\,dx - \frac{D_1}{2}\int_0^L d(x)(\delta_x^2)_t\,dx\\ &-K_1D_1\int_0^L b(x)\epsilon_x\delta_x\,dx - K_1D_1\int_0^L d(x)\delta_x^2\,dx\,. \qquad (44)\end{aligned}$$

Adding (39) and (44), we get,

$$\begin{aligned}
&\left[\int_0^L B(\epsilon)\,dx\right]_t + \left[\int_0^L B(\delta)\,dx\right]_t + \frac{K_1K_2}{2}\left[\int_0^L \delta_x^2\,dx\right]_t \\
&+\frac{D_1}{2}\left[\int_0^L d(x)\delta_x^2\,dx + 2\int_0^L b(x)\delta_x\epsilon_x\,dx + \int_0^L \epsilon_x^2\,dx\right]_t \\
&\quad+\frac{D_2+K_1K_3}{2}\left[\int_0^L \delta_{xx}^2\,dx\right]_t = -K_2\int_0^L \delta_{tx}^2\,dx - K_3\int_0^L \delta_{txx}^2\,dx \\
&\quad -K_1D_1\left[\int_0^L d(x)\delta_x^2\,dx + 2\int_0^L b(x)\delta_x\epsilon_x\,dx + \int_0^L \epsilon_x^2\,dx\right] \\
&\quad -K_1D_2\int_0^L \delta_{xx}^2\,dx\,.
\end{aligned} \tag{45}$$

Let $B_5 = \int_0^L d(x)\delta_x^2\,dx$, $B_6 = 2\int_0^L b(x)\delta_x\epsilon_x\,dx$, $B_7 = \int_0^L \epsilon_x^2\,dx$, and recalling that $d(x) = C + b(x)^2$ one can show that,

$$B_5 + B_6 + B_7 = C\int_0^L \delta_x^2\,dx + \int_0^L (b(x)\delta_x + \epsilon_x)^2\,dx\,. \tag{46}$$

Substituting (46) in (45), we then obtain

$$\begin{aligned}
&\frac{1}{2}\left[\int_0^L (\epsilon_t + K_1\epsilon)^2\,dx\right]_t + \frac{1}{2}\left[\int_0^L (\delta_t + K_1\delta)^2\,dx\right]_t \\
&+\frac{D_1C+K_1K_2}{2}\left[\int_0^L \delta_x^2\,dx\right]_t + \frac{D_2+K_1K_3}{2}\left[\int_0^L \delta_{xx}^2\,dx\right]_t \\
&\quad+\frac{D_1}{2}\left[\int_0^L (\epsilon_x + b(x)\delta_x)^2\,dx\right]_t = -K_2\int_0^L \delta_{tx}^2\,dx - K_3\int_0^L \delta_{txx}^2\,dx \\
&\quad -K_1D_1C\int_0^L \delta_x^2\,dx - K_1D_1\int_0^L (b(x)\delta_x+\epsilon_x)^2\,dx - K_1D_2\int_0^L \delta_{xx}^2\,dx\,.
\end{aligned} \tag{47}$$

Now let us define

$$\begin{aligned}
I(\epsilon,\delta,t) = &\frac{1}{2}\int_0^L (\epsilon_t + K_1\epsilon)^2\,dx + \frac{1}{2}\int_0^L (\delta_t + K_1\delta)^2\,dx \\
&+\frac{D_1C+K_1K_2}{2}\int_0^L \delta_x^2\,dx + \frac{D_2+K_1K_3}{2}\int_0^L \delta_{xx}^2\,dx \\
&+\frac{D_1}{2}\int_0^L (\epsilon_x + b(x)\delta_x)^2\,dx\,.
\end{aligned} \tag{48}$$

All constant terms are nonnegative, therefore $I(\epsilon,\delta,t) \geq 0$, and using the boundary conditions, $I(\epsilon,\delta,t) = 0$ if and only if $\epsilon = \delta = 0$. Taking into account Eq. (47) and definition (48)

$$\begin{aligned}
I'(\epsilon,\delta,t) = &-K_2\int_0^L \delta_{tx}^2\,dx - K_3\int_0^L \delta_{txx}^2\,dx \\
&-K_1D_1C\int_0^L \delta_x^2\,dx - K_1D_1\int_0^L (b(x)\delta_x+\epsilon_x)^2\,dx - K_1D_2\int_0^L \delta_{xx}^2\,dx \leq 0\,.
\end{aligned} \tag{49}$$

That implies that $I(\epsilon,\delta,t)$ is a positive definite function which is not-increasing in time. Thus we proved that:

Theorem 5.3. *The steady state equilibrium is monotonically stable.*

Remark 1. The steady state equilibrium is monotonically stable if at least one of the damping coefficients (K_1, K_2 or K_3) is different from zero.

To prove that the system is also asymptotically stable let us first denote

$$I(\epsilon,\delta,t)=\sum_{i=1}^{5} y_i(\epsilon,\delta,t)\,, \tag{50}$$

where

$$y_1(t)=\alpha_1\int_0^L \delta_x^2\,dx \quad \text{with} \quad \alpha_1=\frac{D_1C+K_1K_2}{2}\,, \tag{51}$$

$$y_2(t)=\alpha_2\int_0^L \delta_{xx}^2\,dx \quad \text{with} \quad \alpha_2=\frac{D_2+K_1K_3}{2}\,, \tag{52}$$

$$y_3(t)=\alpha_3\int_0^L (\epsilon_x+b(x)\delta_x)^2\,dx \quad \text{with} \quad \alpha_3=\frac{D_1}{2}\,, \tag{53}$$

$$y_4(t)=\alpha_4\int_0^L (\delta_t+K_1\delta)^2\,dx \quad \text{with} \quad \alpha_4=\frac{1}{2}\,, \tag{54}$$

$$y_5(t)=\alpha_5\int_0^L (\epsilon_t+K_1\epsilon)^2\,dx \quad \text{with} \quad \alpha_5=\frac{1}{2}\,. \tag{55}$$

We interpret y_i as the i-component of the vector valued function $\vec{y}(t)=(y_1,y_2,...,y_5)$. We will prove by contradiction that any trajectory $\vec{y}(t)$ converges to the origin. Let assume that there exists a trajectory $\vec{y}(t)$ such that

$$\|\vec{y}(t)\|:=I(t)>A>0 \quad \text{for all} \quad t>T\,. \tag{56}$$

According to (49) the derivative along this trajectory is given by

$$\frac{d\|\vec{y}(t)\|}{dt}=\frac{dI(t)}{dt}=\sum_{i=0}^{5}\frac{\partial I}{\partial y_i}\frac{\partial y_i}{\partial t}=-V(\epsilon,\delta)\,, \tag{57}$$

$$\begin{aligned} V(\epsilon,\delta)&=\beta_1\int_0^L \delta_x^2\,dx+\beta_2\int_0^L \delta_{xx}^2\,dx+\\ &+\beta_3\int_0^L (b(x)\delta_x+\epsilon_x)^2\,dx+K_2\int_0^L \delta_{tx}^2\,dx+K_3\int_0^L \delta_{txx}^2\,dx\,, \end{aligned} \tag{58}$$

where $\beta_1=K_1D_1C$, $\beta_2=K_1D_2$, and $\beta_3=K_1D_1$. Inequality (56) implies at least one of the five following cases:

i) $y_1>q_1A$. For convenience the positive constant q_1 will be selected later. In this case $V(\epsilon,\delta)>\frac{\beta_1}{\alpha_1}q_1A=\gamma_1A$ for all $t>T$.

ii) $y_2>q_2A$. For convenience the positive constant q_2 will be selected later. In this case $V(\epsilon,\delta)>\frac{\beta_2}{\alpha_2}q_2A=\gamma_2A$ for all $t>T$.

iii) $y_3>A/5$. In this case $V(\epsilon,\delta)>\frac{\beta_3}{\alpha_3}A/5=\gamma_3A$ for all $t>T$.

iv) $y_4>A/5$. From this and from Cauchy inequality it follows

$$\alpha_4\int_0^L \left((1+K_1)\delta_t^2+(K_1+K_1^2)\delta^2\right)\,dx\geq\alpha_4\int_0^L (\delta_t+K_1\delta)^2\,dx>\frac{A}{5}\,. \tag{59}$$

[*Subcase-a*:] If

$$\alpha_4\int_0^L (K_1+K_1^2)\delta^2\,dx>\frac{A}{10}\,, \tag{60}$$

then from Friedrichs inequality

$$C_F\alpha_4\int_0^L (K_1+K_1^2)\delta_x^2\,dx\geq\alpha_4\int_0^L (K_1+K_1^2)\delta^2\,dx>\frac{A}{10}\,, \tag{61}$$

from which we can conclude that

$$V(\epsilon,\delta)>\frac{\beta_1}{\alpha_4C_F(K_1+K_1^2)}\frac{A}{10}\,, \quad \text{for all} \quad t>T\,. \tag{62}$$

[*Subcase-b*:] Otherwise, it is

$$\alpha_4 \int_0^L (1+K_1)\delta_t^2\, dx > \frac{A}{10}\,. \tag{63}$$

Again, applying Friedrichs inequality

$$C_F\alpha_4 \int_0^L (1+K_1)\delta_{xt}^2\, dx \geq \alpha_4 \int_0^L (1+K_1)\delta_t^2\, dx > \frac{A}{10}\,, \tag{64}$$

from which we can conclude that

$$V(\epsilon,\delta) > \frac{K_2}{\alpha_4 C_F(1+K_1)}\frac{A}{10}\,, \quad \text{for all } t > T\,. \tag{65}$$

As a result we got $V(\epsilon,\delta) > \gamma_4 A$ for all $t > T$, where the constant

$$\gamma_4 = \min\left(\frac{\beta_1}{10\,\alpha_4 C_F(K_1+K_1^2)}, \frac{K_2}{10\,\alpha_4 C_F(1+K_1)}\right)\,. \tag{66}$$

v) $y_5 > A/5$. From this and form Cauchy inequality it follows

$$\alpha_5 \int_0^L ((1+K_1)\varepsilon_t^2 + (K_1+K_1^2)\varepsilon^2)\, dx \geq \alpha_5 \int_0^L (\varepsilon_t + K_1\varepsilon)^2\, dx > \frac{A}{5}\,. \tag{67}$$

[*Subcase-a*:] If

$$\alpha_5 \int_0^L (K_1+K_1^2)\varepsilon^2\, dx > \frac{A}{10}\,, \tag{68}$$

then from Friedrichs inequality

$$C_F\alpha_5 \int_0^L (K_1+K_1^2)\varepsilon_x^2\, dx \geq \alpha_5 \int_0^L (K_1+K_1^2)\varepsilon^2\, dx > \frac{A}{10}\,. \tag{69}$$

Now assume $y_1 < q_1 A$, otherwise we are already in case (i). Then from Cauchy inequality it follows that

$$\int_0^L \left(\epsilon_x + b(x)\delta_x\right)^2\, dx \geq \int_0^L \left(\frac{\varepsilon_x^2}{2} - (b(x)^2+1)\delta_x)^2\right)\, dx\,. \tag{70}$$

From above and from Eq. (69) it follows that

$$\int_0^L \left(\frac{\varepsilon_x^2}{2} - (b(x)^2+1)\delta_x)^2\right) dx > \frac{A}{20C_F\alpha_5(K_1+K_1^2)} - \frac{\max(b(x)^2+1)q_1 A}{\alpha_1}\,. \tag{71}$$

Let

$$q_1 = \frac{\alpha_1}{40\,\alpha_5 C_F(K_1+K_1^2)\max(b(x)^2+1)}\,, \tag{72}$$

then we have

$$\int_0^L (\varepsilon_x + b(x)\delta_x)^2\, dx \geq \frac{A}{40C_F\alpha_5(K_1+K_1^2)}\,, \tag{73}$$

As a result we got $V(\epsilon,\delta) > \gamma_5 A$ for all $t > T$, where γ_5 is a positive constant depending only on coefficients and $\max b(x)^2$.

[*Subcase-b*:] If (68) is not true than we have

$$\alpha_5 \int_0^L (1+K_1)\varepsilon_t^2\, dx > \frac{A}{10}\,. \tag{74}$$

From integrating Eq. (35), integrating by parts and using boundary conditions it follows

$$\frac{1}{2}\left[\int_0^L \epsilon_t^2\, dx + D_1 \int_0^L \epsilon_x^2\, dx\right]_t = -\int_0^L K_1\epsilon_t^2\, dx + D_1 \int_0^L \epsilon_t\,[b(x)\delta_x]_x\, dx\,. \tag{75}$$

By using Cauchy inequality we can assert that

$$D_1\epsilon_t\,[b(x)\delta_x]_x \leq \frac{K_1}{4}\epsilon_t^2 + \frac{D_1^2}{K_1}([b(x)\delta_x]_x)^2\,. \tag{76}$$

From (75) and (76), it follows

$$\frac{1}{2}\left[\int_0^L \epsilon_t^2\,dx + D_1\int_0^L \epsilon_x^2\,dx\right]_t \le -\frac{3}{40}\frac{K_1 A}{\alpha_5(1+K_1)} + \int_0^L \frac{D_1^2}{K_1}([b(x)\delta_x]_x)^2\,dx\,. \tag{77}$$

Expanding the argument of the integral in RHS of above inequality,

$$\big([b(x)\delta_x]_x\big)^2 = b_x(x)^2\delta_x^2 + 2\,b_x(x)\delta_x\,b(x)\delta_{xx} + b(x)^2\delta_{xx}^2$$

and applying Cauchy inequality to the middle term it follows

$$\int_0^L \frac{D_1^2}{K_1}([b(x)\delta_x]_x)^2\,dx \le r_1y_1 + r_2y_2\,, \tag{78}$$

where the positive constant r_1 and r_2 depend only on coefficients and $\max\big(b(x)^2\big)$ and $\max\big(b_x(x)^2\big)$. Now assume $y_1 < q_1A$ and $y_2 < q_2A$, otherwise we are in case (i) or (ii). Let us chose q_1 and q_2 such that

$$(r_1q_1 + r_2q_2) < \frac{1}{40}\frac{K_1}{\alpha_5(1+K_1)}\,. \tag{79}$$

Combining (77), (78) and (79) we obtain

$$\frac{dJ(t)}{dt} := \frac{1}{2}\left[\int_0^L \epsilon_t^2\,dx + D_1\int_0^L \epsilon_x^2\,dx\right]_t \le -\frac{1}{20}\frac{K_1 A}{\alpha_5(1+K_1)}\,. \tag{80}$$

From above constructions it follows the following. If there exists a trajectory $\vec{y}(t)$ which is not converging to the origin then or the functional $V(\epsilon,\delta) > \min(\gamma_i)A = \alpha_0$ for $t > T$ or inequality (80) holds. Therefore along the trajectory $\vec{y}(f)$ or

$$\frac{dI(t)}{dt} < -\alpha_0\,, \quad \text{with } \alpha_0 = \min(\gamma_i)A\,, \tag{81}$$

or

$$\frac{dJ(t)}{dt} < -\beta_0\,, \quad \text{with } \beta_0 = \frac{1}{20}\frac{K_1 A}{\alpha_5(1+K_1)}\,. \tag{82}$$

This implies that $I(t)$ or $J(t)$ becomes negative for big enough t, which contradicts the fact that $I(t)$ and $J(t)$ are positive definite.

Thus we proved that:

Theorem 5.4. *The steady state equilibrium is asymptotically stable.*

6. **Linearized system with perturbed right hand side.** In this section we present some preliminary results of the coupled linearized system around the equilibrium (U,W), with perturbed right hand side q. The aim of this analysis is to indicate some differences which arise when the beam system is coupled with some other physical system, for example fluid motion.

Consider the governing differential equations for the coupled linearized system, with both axial and transverse prescribed loads and with $K_2 = K_3 = 0$,

$$\epsilon_{tt} + K_1\epsilon_t - D_1\epsilon_{xx} - D_1(b(x)\delta_x)_x = f(t)g(x)\,, \tag{83}$$

$$\delta_{tt} + K_1\delta_t - D_1(b(x)\epsilon_x)_x - D_1(d(x)\delta_x)_x + D_2\delta_{xxxx} = f(t)q(x)\,. \tag{84}$$

Denote vector $\vec{u}(x,t) = y(t)\,\vec{w}(x)$, where

$$\vec{w}(x) = \begin{pmatrix} \phi_\lambda(x) \\ \varphi_\lambda(x) \end{pmatrix}$$

is the vector of eigenfunctions for the problem

$$-D_1\phi_{\lambda xx} - D_1[b(x)\varphi_{\lambda x}]_x = \lambda\phi_\lambda\,, \tag{85}$$

$$-D_1[b(x)\phi_{\lambda x}]_x - D_1[d(x)\varphi_{\lambda x}]_x + D_2\varphi_{\lambda xxxx} = \lambda\varphi_\lambda\,, \tag{86}$$

with boundary conditions

$$\phi_\lambda(0) = \phi_\lambda(L) = 0\,, \tag{87}$$

$$\varphi_\lambda(0) = \varphi_\lambda(L) = \varphi_{\lambda x}(0) = \varphi_{\lambda x}(L) = 0\,. \tag{88}$$

Lemma 6.1. *If the solution of the spectral problem (85-88) exists then $\lambda > 0$.*

Proof. Let $f(t) = 0$, then the system (83-84) reduces to the linear homogeneous system analyzed in Section 5. Substituting equations (85-86) in (83-84) yields

$$y_{tt} + K_1 y_t + \lambda y = 0\,. \tag{89}$$

The characteristic polynomial associated with this second order ODE and and its roots are given by

$$r^2 + K_1 r + \lambda = 0 \quad \text{and} \quad r_{1,2} = \frac{-K_1 \pm \sqrt{K_1^2 - 4\lambda}}{2}\,. \tag{90}$$

Clearly, if $\lambda \leq 0$ one of the two roots will be necessary greater or equal than zero. This implies that the norm of the solution $I(\epsilon, \delta, t)$ does not decrease in time, which contradicts the conclusions made in Section 5 about stability of the system. Therefore $\lambda > 0$. □

In Table 1, numerical evaluation of the smallest eigenvalue is reported for a large variety of coefficients D_1 and D_2. All these results are in agreement with Lemma 6.1. In Figure 3 the first three eigenvalues and corresponding eigenfunctions (ϕ, φ) are shown for $D_1 = 1$, $D_2 = 1$, $Q = 1$ and $L = 1$.

In this article we are not interested in the proof of existence of spectral problem but rather in its application.

$D1$	λ
10	98.7
1	9.87
0.1	0.987
0.01	0.0987
0.001	0.00987
0.0001	0.000987

$D2 = 1$

$D2$	λ
10000	9.87
100	9.87
1	9.87
0.01	8.70
0.001	4.96
0.0001	3.85
0.000001	3.55

$D1 = 1$

$D1$	λ
1000	99.49
100	46.08
10	21.1
1	4.96
0.1	0.663
0.01	0.0805
0.001	0.00925
0.0001	0.000977
0.00001	0.0000986

$D2 = 0.001$

TABLE 1. Smallest eigenvalue for different values of the parameters D_1 and D_2 with $Q = 1$ and $L = 1$.

Proposition 1. *If*

$$\vec{w}(x) = \begin{pmatrix} \phi_\lambda(x) \\ \varphi_\lambda(x) \end{pmatrix}$$

is solution of (85-88), and $g = \phi_\lambda$, $q = \varphi_\lambda$, then in order for $\vec{u}(x,t)$ to be a solution of the system (83-84) it is necessary and sufficient that

$$y_{tt} + K_1 y_t + \lambda y = f(t)\,. \tag{91}$$

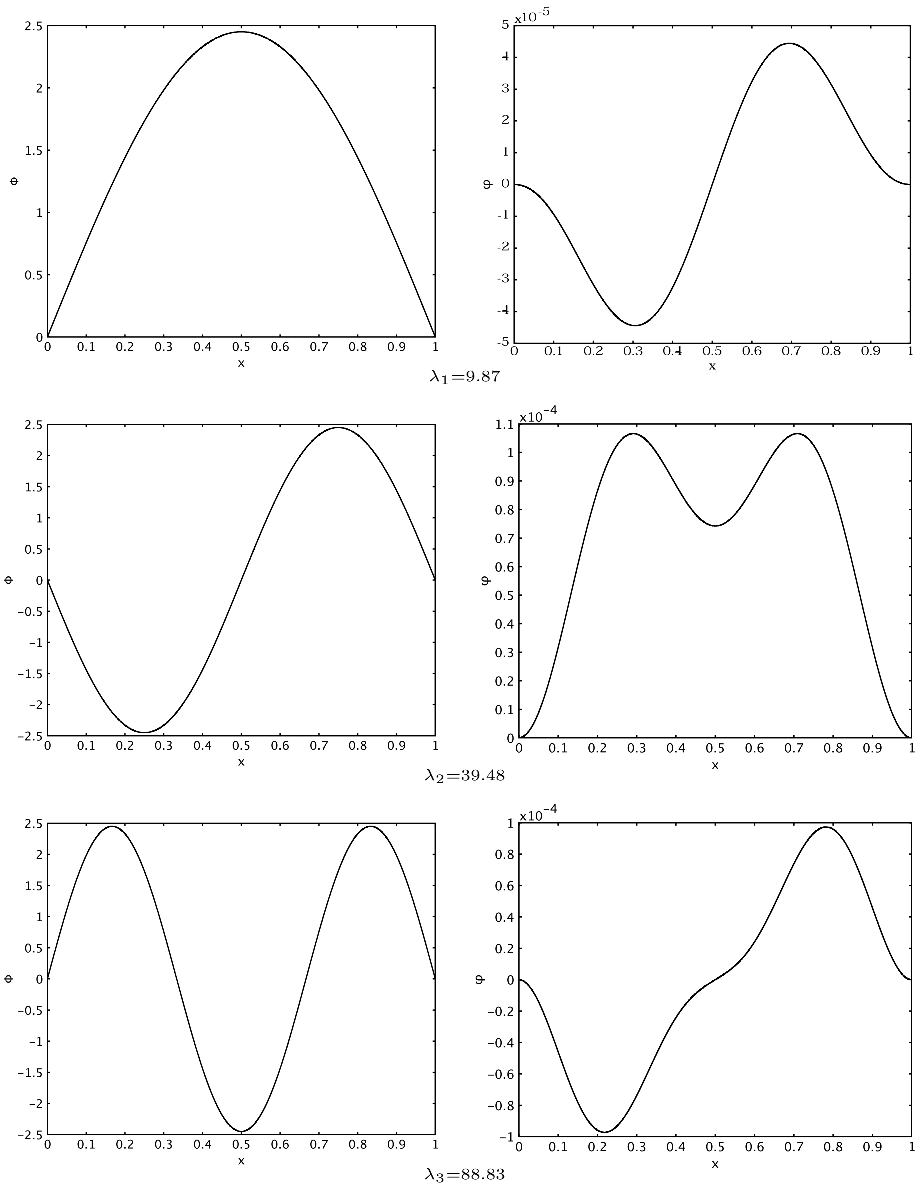

FIGURE 3. First three eigenvalues and corresponding eigenfunctions, for $D_1 = 1$, $D_2 = 1$, $Q = 1$ and $L = 1$.

Assume that Proposition 1 holds, then we can build two simple examples where some sort of instabilities may occur.

Example 1. Let $4\lambda > K_1^2$ and assume

$$f(t) = e^{\alpha t}\cos(\beta t)\,, \tag{92}$$

where

$$\alpha = -\frac{K_1}{2}, \quad \beta = \frac{\sqrt{4\lambda - K_1^2}}{2}\,. \tag{93}$$

Then

$$y(t) = te^{\alpha t}[\frac{1}{2\beta}\sin(\beta t)] \tag{94}$$

solves the equation (91). For any given value of K_1, even when the amplitude of the perturbation of the RHS of Eq. (91) decays exponentially, there exists a time interval $[0, T]$ where $y(t)$ increases. Hence, for this example the steady state solution equilibrium is not monotonically stable. The critical point of the envelope curves $g_{1/2}(t) = \pm te^{\alpha t}$ occurs in $t_{max} = \frac{2}{K_1}$, with maximum amplitude given by $|g_{1/2}(t_{max})| = \frac{2}{eK_1}$. Clearly, depending on the value of K_1, the maximum amplitude can be large. In many physical processes this may already indicate a structural collapse of the beam system.

Example 2. In the next example we show that if the damping coefficient $K_1 = K_1(t)$ decays faster than t,

$$t\,K_1(t) \to 0 \quad \text{as} \quad t \to \infty,$$

then the system

$$y_{tt} + K_1(t)y_t + \lambda y = f(t)\,, \tag{95}$$

may resonate with a bounded RHS. Let

$$f(t) = \cos(\alpha t) + \frac{1}{2\sqrt{\lambda}}K_1(t)(\sin(\alpha t) + \alpha t\cos(\alpha t)), \tag{96}$$

where $\alpha = \sqrt{\lambda}$. Then it is easy to see that

$$y(t) = \frac{t}{2\sqrt{\lambda}}\sin(\alpha t)\,, \tag{97}$$

is solution of Eq. (95) and that the system resonates for any $K_1(t)$. Now, if $t\,K_1(t) \to 0 \quad \text{as} \quad t \to \infty$ than the RHS results to be bounded, and satisfies $f(t) \to \cos(\alpha t) \quad \text{as} \quad t \to \infty$. Clearly, monotonically and asymptotically stability conditions are not satisfied.

7. **Conclusion.** In this work the stability analysis of the steady state equilibrium of a non-linear beam transversely and axially excited has been considered. It has been shown that in the presence of damping terms there exists an appropriate energy norm for which the system is stable near the equilibrium for any perturbation. Different damping terms have been considered: first, third and fifth order mixed derivatives. We proved that the system is monotonically stable if at least one of the damping terms is different from zero. If all the damping coefficients are different than zero then the system is asymptotically stable around equilibrium.

In case of perturbed RHS we have shown that if the solution of a particular eigenvalue system exists then the corresponding smallest eigenvalue is positive. By using this result we built two simple examples where in a certain sense instabilities may occurs. In the first example we showed that by choosing a particular exponentially decaying force there exists a time interval $[0, T]$, for which the solution may increase. In the second one we showed that the solution can resonate if the damping coefficient decays faster that $\frac{1}{t}$, even with a bounded RHS.

REFERENCES

[1] A. A. Alqaisia and M. N. Hamdan, *Bifurcation and chaos of an immersed cantilever beam in a fluid and carrying an intermediate mass*, Journal of Sound and Vibration, **253** (2002), 859–888.

[2] A. Andrianov and A. Hermans, *A VELFP on infinite, finite and shallow water*, 17-th International workshop on water waves and floating bodies, Cambridge, UK, aPEIL (2002), 14–17.

[3] E. Aulisa, A. Ibragimov, Y. Kaya and P. Seshaiyer, *A stability estimate for fluid structure interaction problem with non-linear beam,* Accepted in the "Proceeding of the Seventh AIMS International Conference on Dynamical Systems."

[4] E. Aulisa, A. Cervone, S. Manservisi and P. Seshaiyer, *A multilevel domain decomposition approach for studying coupled flow application,* Communications in Computational Physics, **6** (2009), 319–341.

[5] E. Aulisa, S. Manservisi, and P. Seshaiyer, *A computational domain decomposition approach for solving coupled flow-structure-thermal interaction problems*, Seventh Mississippi State - UAB Conference on Differential Equations and Computational Simulations. Electron. J. Diff. Eqns., Conference 17 (2009), pp. 13–31.

[6] E. Aulisa, S. Manservisi and P. Seshaiyer, *A multilevel domain decomposition methodology for solving coupled problems in fluid-structure-thermal interaction,* Proceedings of ECCM 2006, Lisbon, Portugal (2006).

[7] R. W. Dickey, *Dynamic stability of equilibrium states of the extendible beam*, Proceedings of the American Mathematical Society, **41** (1973), 94–102.

[8] L. C. Evans, "Partial Differential Equations," AMS, 1998.

[9] D. A. Evensen, *Nonlinear vibrations of beams with various boundary conditions*, AIAA Journal, **6** (1968), 370–372.

[10] L. Ferguson, E. Aulisa, P. Seshaiyer, *Computational modeling of highly flexible membrane wings in micro air vehicles*, Proceedings of the 47th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference Newport, RI (2006).

[11] D. G. Gorman, I. Trendafilova, A. J. Mulholland and J. Horacek, *Analytical modeling and extraction of the modal behavior of a cantilever beam in fluid interaction*, Journal of Sound and Vibration, (2007), 231–245.

[12] A. E. Green and J. E. Adkins, "Large Elastic Deformations," Clarendon Press (Oxford), 1970.

[13] H. W. Haslach, J. D. Humphrey, *Dynamics of biological soft tissue or rubber: Internally pressurized spherical membranes surrounded by a fluid,* Int J Nonlin Mech, **39** (2004), 399–420.

[14] J. D. Humphrey, "Cardiovascular Solid Mechanics," Springer, 2002.

[15] A. I. Ibragimov and P. Koola, *The dynamics of wave carpet*, P. 2288, OCEAN 2003 MTS/IEEE, proceedings.

[16] R. A. Ibrahim, *Nonlinear vibrations of suspended cables, Part III: Random excitation and interaction with fluid flow*, Applied Mechanics Reviews, **57** (2004), 515–549.

[17] J. E. Lagnese, *Modelling and stabilization of nonlinear plates*, International Series of Numerical Mathematics, **100** (1991), 247–264.

[18] C. L. Lou and D. L. Sikarskie, *Nonlinear Vibration of beams using a form-function approximation*, ASME Journal of Applied Mechanics, **42** (1975), 209–214.

[19] C. Mei, *Finite element displacement method for large amplitude free flexural vibrations of beams and plates*, Computers and Structures, **3** (1973), 163–174.

[20] J. Padovan, *Nonlinear vibrations of general structures*, Journal of Sound and Vibration, **72** (1980), 427–441.

[21] J. Peradze, *A numerical alghorithm for Kirchhoff-Type nonlinear static beam*, Journal of Applied Mathematics, in Press (2009).

[22] J. N. Reddy, *Finite element modeling of structural vibrations: A review of recent advances*, The Shock Vibration Digest, **11**, 25–39.

[23] J. N. Reddy, *An introduction to Nonlinear Finite Element Analysis,* Oxford University, 2004.

[24] D. L. Russel, *A comparison of certain dissipation mechanisms via decoupling and projection techniques*, Quart. Appl. Math., **XLIX** (1991), 373–396.

[25] P. Seshaiyer and J. D. Humphrey, *A sub-domain inverse finite element characterization of hyperelastic membranes, including soft tissues,* ASME J Biomech Engr., **125** (2003), 363–371.

[26] W. Shyy, Y. Lian, J. Tang, D. Viieru and H. Liu, *Aerodynamics of Low Reynolds Number Flyers,* Cambridge University Press, 2007.

[27] G. Singh, G. V. Rao and N. G. R. Iyengar, *Reinvestigation of large amplitude free vibrations of beams using finite elements*, Journal of Sound and Vibration, **143** (1990), 351–355.

[28] H. Wagner and V. Ramamurti, *Beam vibrations-A review*, The Shock and Vibration Digest, **9** (1977), 17–24.

[29] O. C. Zienkiewicz and R. L. Taylor, "The Finite Element Method," McGraw-Hill, 1993,

Received May 2009; revised November 2010.

E-mail address: yasemen.kaya@ttu.edu
E-mail address: eugenio.aulisa@ttu.edu
E-mail address: akif.ibraguimov@ttu.edu
E-mail address: pseshaiy@gmu.edu