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NON-CONFORMING COMPUTATIONAL METHODS FOR MIXED ELASTICITY PROBLEMS [1]

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Dedicated to Raytcho Lazarov on the occasion of his 60th birthday.

Abstract — In this paper, we present a non-conforming hp computational modeling methodology for solving elasticity problems. We consider the incompressible elasticity model formulated as a mixed displacement-pressure problem on a global domain which is partitioned into several local subdomains. The approximation within each local subdomain is designed using div-stable hp-mixed finite elements. The displacement is computed in a mortared space while the pressure is not subjected to any constraints across the interfaces. Our computational results demonstrate that the non-conforming finite element method presented for the elasticity problem satisfies similar rates of convergence as the conforming finite element method, in the presence of various h-version and p-version discretizations.

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1. Introduction

Many applications in science and engineering often require complicated finite element modeling with robust elements for elasticity problems. Often such design is accomplished by

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first partitioning the global domain into several local subdomains. Each of these local subcomponents is then modeled independently, maybe using different discretizations based on the application. Finally, these individually discretized subcomponents are coupled along the respective interfaces to realize the global domain. In this procedure, the finite element nodes of each subdomain often do not coincide at the common interface. This motivates the necessity for employing non-conforming methods at the subdomain level.

The mortar finite element method [10] is an example of such a non-conforming technique which can be used to de-compose and re-compose a domain into subdomains without requiring compatibility between the meshes on the separate components. This method was first analyzed for the Poisson problem [3, 8, 21, 23–25] and was later extended to the Stokes problem [1, 4–7, 14] for the h, p, hp-versions. In this method, precise choices are described for the two fields (the interior solution variable and the interface Lagrange multiplier). These choices ensure that the method is *stable* i.e., an inf-sup condition is satisfied. Let us mention also the existence of *three-field* non-conforming techniques where one also has a third field on the interface beside the solution variable and the Lagrange multipliers [13, 22]. One can also find other specific formulations such as the mortar finite volume methods [17] and multigrid techniques for mortars [11, 18, 27], in the literature. A discussion on the iterative substructuring method for the mortar finite elements can be found in [15, 16]. These methods are becoming increasingly popular as specialized domain decomposition techniques for treating second-order partial differential equations on any type of domain, with very few restrictions on the grid related to the discretization procedure.

The computational results available in the literature so far are only for the case of Stokes flow, which is the limiting case of the elasticity problem when $\nu \to \frac{1}{2}$. Hence, the purpose of this paper is to present the hp mortar finite element formulation for the mixed elasticity boundary-value problem and for the first time validate the convergence behavior computationally.

The outline of the paper is as follows. In Section 2, we present the model problem and discuss its finite element formulation. The non-conforming hp finite element discretization is described in Section 3. We formulate our model problem as a mixed method for implementation in Section 4. Finally, our computational experiments for the model problem on a L-shaped domain is presented in Section 5. In particular, we recover optimal convergence rates for these techniques in the presence of highly non-quasiuniform geometric meshes. Our results also show that these methods behave as well as conforming finite element methods.

2. Model problem

We consider linear isotropic elasticity under conditions of plain strain and let our domain $\Omega \subset \mathbb{R}^2$ be a polygon subjected to a body force $\mathbf{f}$ and tractions $\mathbf{g}$ on the boundary Γ. Let $0 \leqslant \nu \leqslant \frac{1}{2}$ be the Poisson ratio and E the modulus of elasticity, and define the Lamé constants

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}. \tag{1}$$

Also, let $\mathbf{u} = (u_1, u_2)$ be the displacement and let the linear strain tensor $\varepsilon(\mathbf{u})$ be given by

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Then the variational form of our problem is: find $\mathbf{u} \in V$ such that for all $\mathbf{v} \in V$

$$2\ \mu(\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{L_2(\Omega)^4} + \lambda(\text{div }\mathbf{u}, \text{div }\mathbf{v})_{L_2(\Omega)} = F(\mathbf{v}), \tag{2}$$

where

$$F(\mathbf{v}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}\ dx + \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{v}\ ds$$

and $V = \mathbf{H}^1(\Omega)$ is the Sobolev space of vector-valued functions with one generalized derivative [2]. We assume that $F(\mathcal{R}) = 0$ for any rigid body motion $\mathcal{R}$. This ensures that (2) has a unique solution modulo rigid body motions (which are assumed eliminated in computations by suitable constraints).

2.1. Standard finite element formulation

Given a sequence of finite element subspaces $\{V_N\}, V_N \subset V$, we define the standard finite element approximation to (2) as: find $\mathbf{u}_N \in V_N$ such that for all $\mathbf{v} \in V_N$

$$2\ \mu(\varepsilon(\mathbf{u}_N), \varepsilon(\mathbf{v}))_{L_2(\Omega)^4} + \lambda(\text{div }\mathbf{u}_N, \text{div }\mathbf{v})_{L_2(\Omega)} = F(\mathbf{v}).$$

We identify the parameter N as the dimension (or the number of degrees of freedom) of the subspace V_N.

2.2. Mixed finite element formulation

We define the new independent unknown $p = -\lambda$ div $\mathbf{u}$ which is a multiple of the sum of the normal stresses, $\sigma_x + \sigma_y$. (Note that as $\lambda \to \infty$, this corresponds to the pressure in the limiting Stokes equations). Then (2) can be written in the Hermann variational form [19]: find $(\mathbf{u}, p) \in V \times W$ such that for all $(\mathbf{v}, q) \in V \times W$

$$\begin{aligned} 2\ \mu(\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{L_2(\Omega)^4} - (p, \text{div }\mathbf{v})_{L_2(\Omega)} &= F(\mathbf{v}), \\ (\text{div }\mathbf{u}, q)_{L_2(\Omega)} + \frac{1}{\lambda}(p, q)_{L_2(\Omega)} &= 0, \end{aligned} \tag{3}$$

where $W = L_2(\Omega)$.

For the mixed finite element method, we assume that we are given a sequence of finite element subspaces $(V_N \times W_N) \subset (V \times W)$. Then our problem becomes finding $(\mathbf{u}_N, p_N) \in (V_N \times W_N)$ such that, for all $(\mathbf{v}, q) \in (V_N \times W_N)$

$$\begin{aligned} 2\ \mu(\varepsilon(\mathbf{u}_N), \varepsilon(\mathbf{v}))_{L_2(\Omega)^4} - (p_N, \text{div }\mathbf{v})_{L_2(\Omega)} &= F(\mathbf{v}), \\ (\text{div }\mathbf{u}_N, q)_{L_2(\Omega)} + \frac{1}{\lambda}(p_N, q)_{L_2(\Omega)} &= 0. \end{aligned}$$

As is well known [12], the accuracy of this method will not only depend upon how well V_N, W_N *approximate* V, W respectively, but also on the *stability* of the pairs (V_N, W_N), i.e., the following inf-sup condition satisfied by them given by

$$\inf_{q \in W_N} \sup_{\mathbf{v} \in \mathbf{V}_N} \frac{b(\mathbf{v}, q)}{||\mathbf{v}||_{\mathbf{H}^1(\Omega)}\ ||q||_{L^2(\Omega)}} \geqslant \alpha > 0, \tag{4}$$

where the bilinear form

$$b(\mathbf{v}, q) = -(\text{div }\mathbf{v}, q)_{L_2(\Omega)}.$$

[2] $H^k(\Omega)$ will denote the usual Sobolev space of functions with k generalized derivatives on Ω, with $L_2(\Omega) = H^0(\Omega)$. $\mathbf{H}^1(\Omega)$ will be its vector analogue. The norm of $H^k(\Omega)$ will be denoted by $||.||_k$

3. Non-conforming hp finite element discretization

We proceed by partitioning the domain Ω into S non-overlapping polygonal subdomains $\{\Omega_i\}_{i=1}^S$, which are geometrically conforming by which we mean that $\partial\Omega_i \cap \partial\Omega_j$ $(i < j)$ is either empty, a vertex, or a collection of entire edges of Ω_i and Ω_j. In the latter case, we denote this interface as Γ_{ij} $(i < j)$ and this will consist of individual common edges γ, $\gamma \subset \Gamma_{ij}$. Let us then define the interface set Γ to be the union of the interface intersections $\partial\Omega_i \cap \partial\Omega_j$ $(i < j)$, which result in a non-empty Γ_{ij}. We further subdivide Ω_i into triangles and parallelograms by *regular* families of meshes $\{\mathcal{T}_h^i\}$. Let the maximum size of the triangulation of subdomain Ω_i be h_i. Also let $h = \max_{1\leqslant i\leqslant S}\{h_i\}$. Note that the triangulations over different Ω_i are independent of each other, with no compatibility enforced across interfaces.

In this formulation only the displacement will be computed in a constrained space while the pressure is not subjected to any particular continuity constraints. For $K \subset \mathbb{R}^2$ and $k \geqslant 0$ integer, let $\mathcal{P}_k(K)$ denote the set of polynomials of total degree $\leqslant k$ on K while $\mathcal{Q}_k(K)$ denotes the set of polynomials of degree $\leqslant k$ in each variable. Denote $\mathbf{Q}_k(K) = \mathcal{Q}_k(K) \times \mathcal{Q}_k(K)$. Let $\mathbf{k}$ be a degree vector, $\mathbf{k} = \{k_1, k_2, \ldots, k_S\}$ which specifies the degree used over each subdomain and denote $k = \min_{1\leqslant i\leqslant S}\{k_i\}$.

Let us now assume that the following local families of piecewise polynomial displacement and pressure spaces are given on Ω_i:

$$\mathbf{V}_{h,k_i}^i = \{\mathbf{u} \in \mathbf{H}^1(\Omega_i) \mid \mathbf{u}|_K \in \mathbf{Q}_k(K) \ \text{ for } \ K \in \mathcal{T}_h^i, \quad \mathbf{u} = 0 \ \text{ on } \partial\Omega_i \cap \partial\Omega\},$$
$$W_{h,k_i}^i = \{q \in L^2(\Omega_i) \mid q|_K \in \mathcal{P}_{k-1}(K)\}.$$

The combination $\mathbf{Q}_k/\mathcal{P}_{k-1}$ has been shown to be uniformly divergence stable by Bernardi and Maday in [9]. In addition Stenberg and Suri have identified several other stable pairs [26].

Definition 3.1. *We now define a non-conforming space*

$$\tilde{\mathbf{V}}_{h,\mathbf{k}} = \{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \mathbf{u}|_{\Omega_i} \in \mathbf{V}_{h,k_i}^i\}.$$

Note that $\tilde{\mathbf{V}}_{h,\mathbf{k}} \not\subset \mathbf{H}^1(\Omega)$ and hence cannot be used for finite element calculations.

We now define two separate trace meshes on Γ_{ij}, one from Ω_i and the other from Ω_j (since the meshes $\mathcal{T}_h^i$ are not assumed to conform across interfaces). In addition to the meshes, the polynomial degrees may also be different across interfaces. Given $\mathbf{u} \in \tilde{\mathbf{V}}_{h,\mathbf{k}}$, we denote the traces of $\mathbf{u}$ on Γ_{ij} from each of the domains Ω_i and Ω_j by $\mathbf{u}^i$ and $\mathbf{u}^j$, respectively. Then we can define the global non-conforming displacement space to be

$$\mathbf{V}_{h,\mathbf{k}} = \left\{\mathbf{u} \in \tilde{\mathbf{V}}_{h,\mathbf{k}} \mid \int_\gamma (\mathbf{u}^i - \mathbf{u}^j)\ \chi\, ds = 0 \ \ \forall \chi \in \mathbf{S}_{h,\mathbf{k}}^{\gamma,ij} \quad \forall \gamma \subset \Gamma_{ij} \subset \Gamma\right\},$$

where $\mathbf{S}_{h,\mathbf{k}}^{\gamma,ij}$ is a space of Lagrange multipliers for each edge $\gamma \subset \Gamma_{ij}$. (Note that $\mathbf{V}_{h,\mathbf{k}} \subset \tilde{\mathbf{V}}_{h,\mathbf{k}}$ and it enforces the inter-domain continuity in a *weak* sense). In the *mortar finite element method* (see References [3,8,10,24] and the references therein) the Lagrange multiplier space $\mathbf{S}_{h,\mathbf{k}}^{\gamma,ij}$ is defined in the following way. Let the mesh $\mathcal{T}_h^i$ induce a mesh $\mathcal{T}_h^i(\Gamma_{ij})$ on Γ_{ij}. Let $\gamma \subset \Gamma_{ij}$ and denote the subintervals of this mesh on γ by $I_l,\ 0 \leqslant l \leqslant M$.

Definition 3.2. *Let,*

$$S_{h,\mathbf{k}}^{\gamma,ij} = \{\chi \in C(\gamma) \mid \chi|_{I_l} \in \mathcal{P}_{k_i}(I_l),\ l = 1, \ldots, M-1; \quad \chi|_{I_l} \in \mathcal{P}_{k_i-1}(I_l),\ l = 0, M\}.$$

Then we set the Lagrange multiplier space to be $\mathbf{S}^{\gamma,ij}_{h,\mathbf{k}} = S^{\gamma,ij}_{h,\mathbf{k}} \times S^{\gamma,ij}_{h,\mathbf{k}}$. Note that imposing the mesh and degree on $S^{\gamma,ij}_{h,\mathbf{k}}$ from the domain Ω_i as has been done here is quite arbitrary, and these can be taken from the domain Ω_j as well, without changing the results obtained. More choices for the Lagrange multiplier space can be found in [25].

The global pressure space is given by

$$W_{h,\mathbf{k}} = \left\{ q \in L^2_0(\Omega) \;\middle|\; q|_{\Omega_i} \in W^i_{h,k_i} \right\}.$$

This space is provided with the $L^2(\Omega)$ - norm while the global displacement space is endowed with a discrete Hilbertian broken norm

$$||\mathbf{u}||^2_* = \sum_{i=1}^{S} ||\mathbf{u}||^2_{\mathbf{H}^1(\Omega_i)}.$$

Note that the spaces $W_{h,\mathbf{k}}$ and $\mathbf{V}_{h,\mathbf{k}}$ then satisfy the following approximation results [2, 7]:

Lemma 3.1. *For all $q \in L^2(\Omega)$ with $q_i = q|_{\Omega_i} \in H^l(\Omega_i)$ and for $\nu = \min(l,k)$, we have*

$$\inf_{q_{h,\mathbf{k}} \in W_{h,\mathbf{k}}} ||q - q_{h,\mathbf{k}}||_{L^2(\Omega)} \leqslant C \sum_{i=1}^{S} \frac{h_i^\nu}{k_i^l} ||q_i||_{H^l(\Omega_i)}. \tag{5}$$

Lemma 3.2. *Let $\mathbf{v} \in \mathbf{H}^1(\Omega)$ with $\mathbf{v}_i = \mathbf{v}|_{\Omega_i} \in \mathbf{H}^{l+1}(\Omega_i)$, $l > \frac{1}{2}$. Then for $\nu = \min(l,k)$*

$$\inf_{\mathbf{v}_{h,\mathbf{k}} \in \mathbf{V}_{h,\mathbf{k}}} ||\mathbf{v} - \mathbf{v}_{h,\mathbf{k}}||_* \leqslant C \sum_{i=1}^{S} \frac{h_i^\nu}{k_i^l} \; |\log k_i|^{\frac{1}{2}} \; ||\mathbf{v}_i||_{\mathbf{H}^{l+1}(\Omega_i)}. \tag{6}$$

The mortar *hp* finite element discretization to (2) is then given as follows:
find $(\mathbf{u}_{h,\mathbf{k}}, p_{h,\mathbf{k}}) \in \mathbf{V}_{h,\mathbf{k}} \times W_{h,\mathbf{k}}$ satisfying

$$\begin{aligned} a_S(\mathbf{u}_{h,\mathbf{k}}, \mathbf{v}) + b_S(\mathbf{v}, p_{h,\mathbf{k}}) &= (\mathbf{f}, \mathbf{v}), \\ b_S(\mathbf{u}_{h,\mathbf{k}}, q) - \frac{1}{\lambda}(p_{h,\mathbf{k}}, q) &= 0, \end{aligned} \tag{7}$$

where the bilinear forms

$$\begin{aligned} a_S(\mathbf{u}, \mathbf{v}) &= 2\mu \sum_{i=1}^{S} (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{L_2(\Omega_i)^4}, \\ b_S(\mathbf{v}, q) &= -\sum_{i=1}^{S} (\text{ div } \mathbf{v}, q)_{L_2(\Omega_i)}. \end{aligned}$$

The existence and uniqueness for this problem can be shown by proving continuity and coercivity of the bilinear forms $a_S(\cdot,\cdot)$ and $b_S(\cdot,\cdot)$. The problem (7) has a unique solution if the following discrete *inf-sup* condition holds (see [7] for more details): There exists a constant α' such that

$$\inf_{q_{h,\mathbf{k}} \in W_{h,\mathbf{k}}} \; \sup_{\mathbf{v}_{h,\mathbf{k}} \in \mathbf{V}_{h,\mathbf{k}}} \frac{b_S(\mathbf{v}_{h,\mathbf{k}}, q_{h,\mathbf{k}})}{||\mathbf{v}_{h,\mathbf{k}}||_* \; ||q_{h,\mathbf{k}}||_{L^2(\Omega)}} \geqslant \alpha' > 0. \tag{8}$$

From (5), (6) and (8) we then have the following global convergence error estimate:

Theorem 3.1. *Let the exact solution* $(\mathbf{v}, q) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ *satisfy* $\mathbf{v}_i = \mathbf{v}|_{\Omega_i} \in \mathbf{H}^{l+1}(\Omega_i)$ *and* $q_i = q|_{\Omega_i} \in H^l(\Omega_i)$ *for* $i = 1, \cdots, S$. *Then for* $\nu = \min(l, k_i)$ *and* α' *given by (8) the discrete solution satisfies*

$$||\mathbf{v}-\mathbf{v}_{h,\mathbf{k}}||_* + \alpha(\lambda)\ ||q-q_{h,\mathbf{k}}||_{L^2(\Omega)} \leqslant C \sum_{i=1}^{S} \frac{h_i^\nu}{k_i^l} \left(|\log k_i|^{\frac{1}{2}}\ ||\mathbf{v}_i||_{\mathbf{H}^{l+1}(\Omega_i)} + \alpha(\lambda)\ ||q_i||_{H^l(\Omega_i)} \right). \quad (9)$$

The precise choise for $\alpha(\lambda)$ is derived in [7]. We then have the following estimates for pure h-version and p-version.

Corollary 3.1. *For a fixed polynomial degree* $k_i = k$ *in (9), the following rate of convergence for the pure h-version holds:*

$$||\mathbf{v} - \mathbf{v}_h||_* + \alpha(\lambda)\ ||q - q_h||_{L^2(\Omega)} = O\left(h^{\min(l,k)}\right).$$

Note that for the non-conforming method, we get a rate of $O(h^k)$ when the solution is smooth enough. Our numerical experiments presented in the next section clearly exhibit this convergence behavior.

Corollary 3.2. *For h fixed in (9), the following p-version estimate holds:*

$$||\mathbf{v} - \mathbf{v}_\mathbf{k}||_* + \alpha(\lambda)\ ||q - q_\mathbf{k}||_{L^2(\Omega)} = O\left(k_i^{-l} |\log k_i|^{\frac{1}{2}}\right).$$

Although the estimate is *quasi-optimal* by the pollution term $\sqrt{|\log k_i|}$, it is very useful in practice for hp computations.

4. Mixed formulation with mortars

Consider the L-shaped domain in Fig. 1. It is somewhat cumbersome to implement the non-conforming method (7) due to the constraints

$$\int_\gamma (\mathbf{u}^a - \mathbf{u}^b)\ \chi\, ds = 0 \quad \forall \chi \in \mathbf{S}_{h,\mathbf{k}}^\gamma.$$

Let us now present a mixed formulation that can be viewed as a convenient method of practically implementing (7). In fact, this has been implemented in Section 5 to perform our numerical experiments. Let us define

$$\sigma_{ij} = \lambda \operatorname{div} \mathbf{u}\ \delta_{ij} + 2\ \mu\ \varepsilon_{ij}(u).$$

Our problem then becomes solving for $\mathbf{u}^r$, $r = a, b$ that satisfy

$$\begin{aligned} -\sigma_{ij,j}^r &= f_i^r && \text{in} \quad \Omega_r, \\ p^r &= -\lambda \operatorname{div} u^r && \text{in} \quad \Omega_r, \\ u_i^r &= 0 && \text{on} \quad \partial\Omega_r^D/\Gamma, \\ \frac{\partial u_i^r}{\partial n} &= g_i^r && \text{on} \quad \partial\Omega_r/(\Gamma \cup \partial\Omega_r^D) = \partial\Omega_r^N, \end{aligned} \quad (10)$$

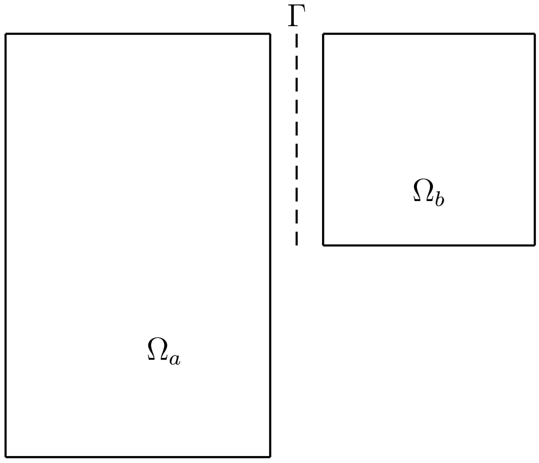

Figure 1. L-shaped domain divided into Ω_a and Ω_b with interface Γ

where $i, j \in 1, 2$. Here $\partial\Omega_r^D$ is the portion of $\partial\Omega_r$ where zero Dirichlet boundary conditions are specified and $\partial\Omega_r^N$ is the portion of $\partial\Omega_r$ where Neumann boundary conditions are specified. These equations are to be solved in conjunction with the continuity condition enforced on the trace of the solution on Γ given by $u_i^a = u_i^b$ on Γ. Let us now define the Lagrange multipliers (for $r = a, b$) to be

$$\lambda_1^r = -(\sigma_{11}^r n_1 + \sigma_{12}^r n_2), \quad \lambda_2^r = -(\sigma_{21}^r n_1 + \sigma_{22}^r n_2).$$

The weak variational form of (10) in the finite dimensional setting then becomes: *find the displacements* $\mathbf{u}^r = (u_1^r, u_2^r) \in \mathbf{V}_N^r \subset \mathbf{H}_0^1(\Omega_r)$, *the pressure* $p^r \in W_N^r \subset L^2(\Omega)$ *and the Lagrange multipliers* $\lambda^r = (\lambda_1^r, \lambda_2^r) \in \mathbf{S}_N^r \subset (H_{00}^{\frac{1}{2}}(\Gamma))'$ *such that,*

$$\begin{aligned}
2\,\mu \int_{\Omega_a} u_{1,1}^a\, v_{1,1}^a\, dx + \mu \int_{\Omega_a} (u_{1,2}^a + u_{2,1}^a)\, v_{1,2}^a\, dx - \int_{\Omega_a} p^a\, v_{1,1}^a\, dx + \int_{\Gamma} \lambda_1\, v_1^a\, ds \\
= \int_{\Omega_a} f_1^a\, v_1^a dx - \int_{\partial\Omega_a^N} p^a\, n_1\, v_1^a\, ds + \int_{\partial\Omega_a^N} (2\mu u_{1,1}^a n_1 + \mu(u_{1,2}^a + u_{2,1}^a) n_2)\, v_1^a\, ds, \\
2\,\mu \int_{\Omega_a} u_{2,2}^a\, v_{2,2}^a\, dx + \mu \int_{\Omega_a} (u_{1,2}^a + u_{2,1}^a)\, v_{2,1}^a\, dx - \int_{\Omega_a} p^a\, v_{2,2}^a dx + \int_{\Gamma} \lambda_2\, v_2^a\, ds \\
= \int_{\Omega_a} f_2^a\, v_2^a dx - \int_{\partial\Omega_a^N} p^a\, n_2\, v_2^a\, ds + \int_{\partial\Omega_a^N} (\mu(u_{2,1}^a + u_{1,2}^a) n_1 + 2\mu u_{2,2}^a n_2)\, v_2^a\, ds, \\
2\,\mu \int_{\Omega_b} u_{1,1}^b\, v_{1,1}^b\, dx + \mu \int_{\Omega_b} (u_{1,2}^b + u_{2,1}^b)\, v_{1,2}^b\, dx - \int_{\Omega_b} p^b\, v_{1,1}^b dx - \int_{\Gamma} \lambda_1\, v_1^b\, ds \\
= \int_{\Omega_b} f_1^b\, v_1^b dx - \int_{\partial\Omega_b^N} p^b\, n_1\, v_1^b\, ds + \int_{\partial\Omega_b^N} (2\mu u_{1,1}^b n_1 + \mu(u_{1,2}^b + u_{2,1}^b) n_2)\, v_1^b\, ds,
\end{aligned}$$

$$2\,\mu \int_{\Omega_b} u^b_{2,2}\, v^b_{2,2}\, dx + \mu \int_{\Omega_b} (u^b_{1,2} + u^b_{2,1})\, v^b_{2,1}\, dx - \int_{\Omega_b} p^b\, v^b_{2,2} dx - \int_{\Gamma} \lambda_2\, v^b_2\, ds$$

$$= \int_{\Omega_b} f^b_2\, v^b_2\, dx - \int_{\partial\Omega^N_b} p^b\, n_2\, v^b_2\, ds + \int_{\partial\Omega^N_b} (\mu(u^b_{2,1} + u^b_{1,2})n_1 + 2\mu u^b_{1,1} n_2)\, v^b_2\, ds,$$

$$\int_{\Omega_a} \operatorname{div} u^a\, q^a\, dx + \frac{1}{\lambda} \int_{\Omega_a} p^a\, q^a\, dx = 0, \qquad \int_{\Omega_b} \operatorname{div} u^b\, q^b\, dx + \frac{1}{\lambda} \int_{\Omega_b} p^b\, q^b\, dx = 0,$$

$$\int_{\Gamma} u^a_1\, \psi_1\, ds - \int_{\Gamma} u^b_1\, \psi_1\, ds = 0, \qquad \int_{\Gamma} u^a_2\, \psi_2\, ds - \int_{\Gamma} u^b_2\, \psi_2\, ds = 0,$$

where we have used the fact that the solution is smooth in the interior and therefore we set $\lambda^a_1 = -\lambda^b_1 = \lambda_1$ and $\lambda^a_2 = -\lambda^b_2 = \lambda_2$.

As the next natural step in the finite element procedure, we define basis functions for each of the finite dimensional spaces $\mathbf{V}^r_N, W^r_N, \mathbf{S}^r_N$ and express the unknowns $\mathbf{u}, p, \lambda$ as a linear combination of the respective basis functions. Choosing the test functions to be basis functions themselves then convert the above system of integral equations into a matrix system that is solved for the unknowns $\mathbf{u}, p, \lambda$.

5. Numerical results

In this section, we investigate the computational performance of the non-conforming method introduced in this paper for both h and p refinements. We used the *mixed* form described in the last section to implement the model problem.

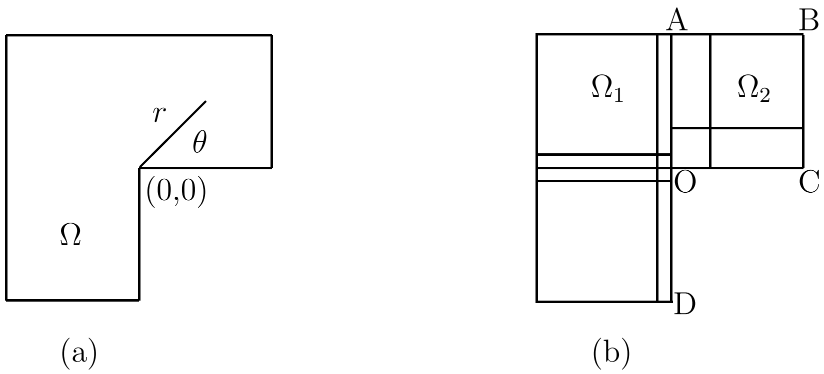

Figure 2. (a) L-shaped domain (b) Tensor product mesh for $m = n = 2$

For our model problem, we let our domain be the L-shaped domain Ω, shown in Fig. 2. This domain is subdivided into two rectangular subdomains Ω_1 and Ω_2 by the interface AO. In our experiments, we consider *tensor product meshes*, where Ω_2 is divided into n^2 rectangles and Ω_1 is divided into $2m^2$ rectangles (see Fig. 2). Since the mesh on Ω_1 is symmetric about $y = 0$, in the sequel we only describe the mesh on the top half.

We solve (2) or (3) with $\mathbf{f} = 0$ and tractions $\mathbf{g}$ specified on the boundary by

$$\begin{aligned} g_1(x,y) &= \frac{8(x-x_0)(y-y_0)\left[(y-y_0)^2-(x-x_0)^2\right]}{\left[(x-x_0)^2+(y-y_0)^2\right]^3} n_2 \\ &+ \frac{4(x-x_0)^2\left[3(y-y_0)^2-(x-x_0)^2\right]}{\left[(x-x_0)^2+(y-y_0)^2\right]^3} n_1, \\ g_2(x,y) &= \frac{8(x-x_0)(y-y_0)\left[(y-y_0)^2-(x-x_0)^2\right]}{\left[(x-x_0)^2+(y-y_0)^2\right]^3} n_1 \\ &+ \frac{4(y-y_0)^2\left[(y-y_0)^2-3(x-x_0)^2\right]}{\left[(x-x_0)^2+(y-y_0)^2\right]^3} n_2. \end{aligned}$$

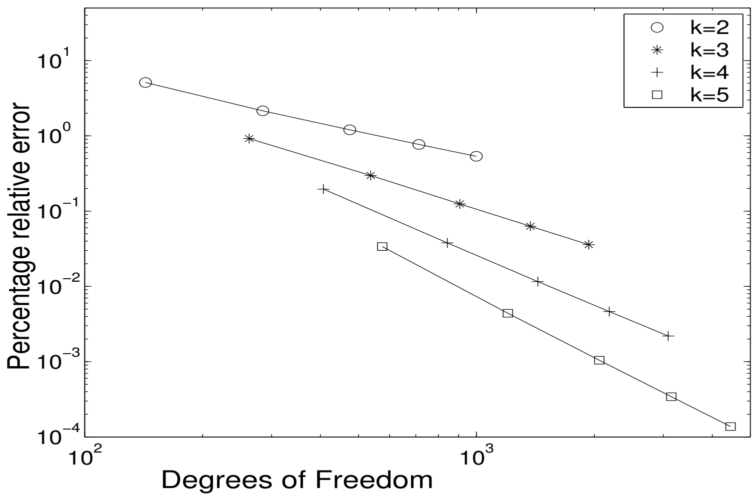

Figure 3. *h*-version convergence for displacement over uniform meshes

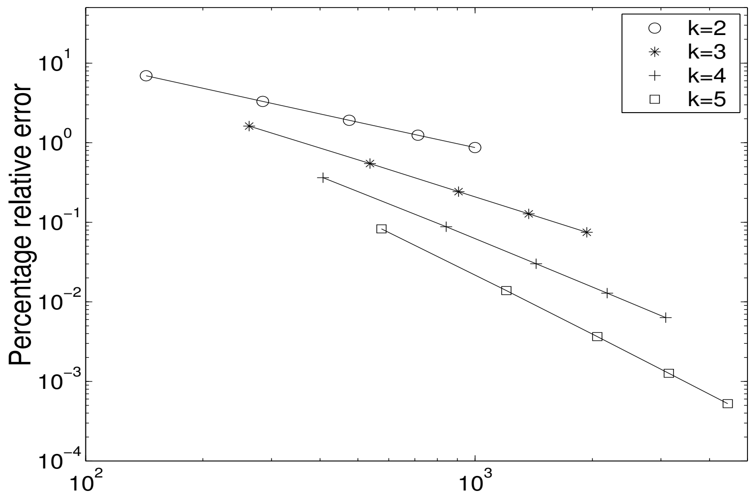

Figure 4. *h*-version convergence for pressure over uniform meshes

Here (n_1, n_2) is the outward unit normal on $\partial\Omega$. The exact solution is given by [20]

$$u_1(x,y) = \frac{(x-x_0)\left[(\lambda+2\mu)(x-x_0)^2 - \lambda(y-y_0)^2\right]}{\mu(\lambda+\mu)\left[(x-x_0)^2+(y-y_0)^2\right]^2},$$
$$u_2(x,y) = \frac{(y-y_0)\left[\lambda(x-x_0)^2 - (\lambda+2\mu)(y-y_0)^2\right]}{\mu(\lambda+\mu)\left[(x-x_0)^2+(y-y_0)^2\right]^2}.$$

Note that λ and μ are defined in terms of E and ν by (1). We take $E = 1$ and $\nu = 0.3$ for our computations. Also, (x_0, y_0) is a point outside Ω.

First we consider the h-version of the non-conforming method on *uniform* meshes. We let $(x_0, y_0) = (1.0, -1.0)$ which yields the case of a *smooth* solution. We take $(m, n) = \{(2,2), (3,3), \cdots, (6,6)\}$ for fixed polynomial degrees $k = 2, 3, 4, 5$. Figures 3 and 4 clearly demonstrate $O(h^k)$ convergence rate as expected in Corollary 3.1.

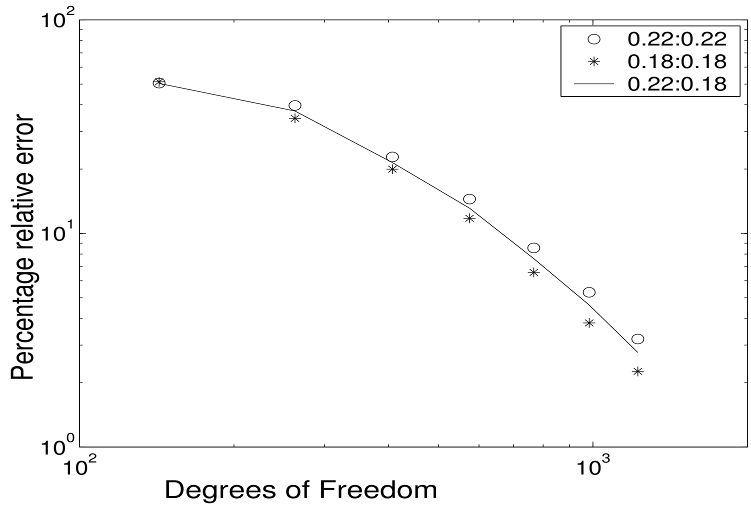

Figure 5. p-version convergence for displacement over geometric meshes

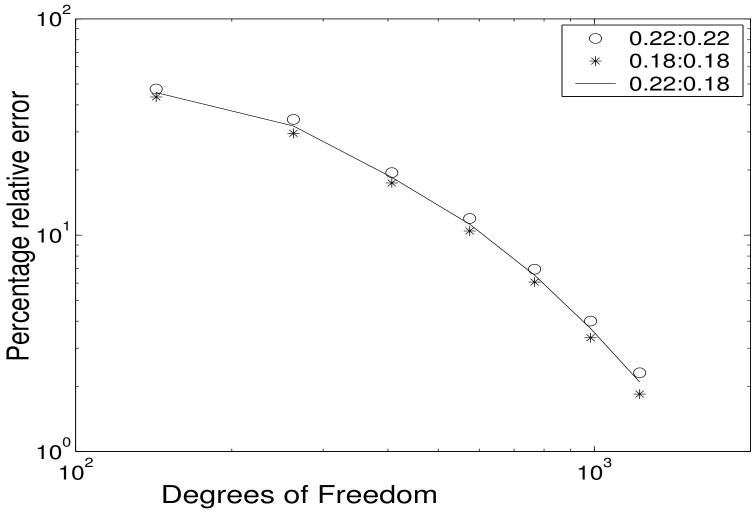

Figure 6. p-version convergence for pressure over geometric meshes

Next we let $(x_0, y_0) = (0.1, -0.1)$. Note that this will yield a *near-singular* solution. We take $m = n = 2$, and along the x and y axes, take the grid points

$$x_0 = 0, \quad x_j = \sigma_i^{n-j}, \quad j = 1, \ldots, n,$$

where σ_i is the geometric ratio used on Ω_i. For this experiment, we consider the combinations $(\sigma_1, \sigma_2) = \{(0.22, 0.22), (0.18, 0.18), (0.18, 0.22)\}$. Note that the first two choices yield a conforming method while the last choice makes the method non-conforming. We increase the polynomial degree $k = 2, \cdots, 8$ to improve the accuracy. The results for the displacement and pressure are illustrated in Figures 5 and 6 respectively. These figures clearly demonstrate that we not only get good convergence rates but also that the convergence rates obtained by employing the non-conforming method $\{(0.18, 0.22)\}$ do not deteriorate and are not any worse than those obtained from the conforming methods $\{(0.22, 0.22), (0.18, 0.18)\}$.

The numerical results presented are in good agreement with the theoretical results and clearly indicate that the mortar finite element method is a robust and viable domain decomposition technique for the elasticity problem. One can extend the current study to materials that are almost incompressible (Lamé constant λ close to ∞, i.e., Poisson ratio ν close to 0.5) and consider the effects of locking. This aspect will be the focus of a following paper.

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