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The hp-Mortar Finite-Element Method for the Mixed Elasticity and Stokes Problems

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Abstract—The motivation of this work is to apply the hp-version of the mortar finite-element method to the nearly incompressible elasticity model formulated as a mixed displacement-pressure problem as well as to Stokes equations in primal velocity-pressure variables. Within each subdomain, the local approximation is designed using div-stable hp-mixed finite elements. The displacement is computed in a mortared space, while the pressure is not subjected to any constraints across the interfaces. By a Boland-Nicolaides argument, we prove that the discrete saddle-point problem satisfies a Babuška-Brezzi inf-sup condition. The inf-sup constant is optimal in the sense that it depends only on the local (to the subdomains) characteristics of the mixed finite elements and, in particular, it does not increase with the total number of the subdomains. The consequences, that we are aware of, of such an important result are twofold.

- The numerical analysis of the approximability properties of the hp-mortar discretization for the mixed elasticity problem allows us to derive an asymptotic rate of convergence that is optimal up to $\sqrt{\log p}$ in the displacement; this is addressed in the present contribution.
- When the mortar discrete problem is inverted by substructured iterative methods based on Krylov subspaces with block preconditioners, in view of the results for conforming finite elements [1], the condition number of the solver should grow logarithmically on (p, h) and not depend on the total number of the subdomains.

1. INTRODUCTION AND NOTATIONS

It is widely admitted that the hp-version of the finite-element method is well suited for the numerical simulation of the solutions of elliptic partial differential equations like the Poisson, Stokes, or elasticity problem set on nonsmooth domains (see [2–4]). A particular design of the meshes (e.g., geometrical meshes) may achieve a convergence of the computed solution towards

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the exact one exponentially fast with respect to the total number of the degrees of freedom (see [5–7]).

Using the *hp*-technique with the mortaring method (see [8]) allows us to handle local conforming/global nonconforming meshes, which highly increases the flexibility of the approximation and then reduces the complexity of the engineering work. A complex domain may be broken up into several smaller components, taking into account some objective criteria (e.g., the local solution's behavior, the availability of efficient solvers on simply shaped subdomains). Each of them is meshed independently and the grids are matched together by mortaring projections without reducing the performances of the local discretizations. Another useful advantage, especially important, of the mortar methods involves the adaptivity. That, for some reason, a refinement is needed somewhere in the computational domain, the possibility offered by the mortar procedure to handle nonmatching grids may limit the contamination as it may be stopped against the interfaces of the decomposition making the refining process confined to only a small region (maybe only one small subdomain).

The original work on the *hp*-mortar finite-element method was developed by Seshaiyer and Suri (see [9]) for the Poisson problem where they piece together the discrete solution in the way it was done for the *h*-version (see [8,10]). The numerical analysis carried out in [9] led to a suboptimal (by $p^{3/4}$) error estimate far from optimality, while computational evidences did not detect any significant deterioration of the accuracy compared to the expected optimal rate (see [11]). In [12], a bootstrapping argument applied to the continuous and discrete linear resolvents of the Poisson problem improved the convergence estimate as the polluting term is reduced to p^{ε}. However, such a technique could fail in some interesting situations. Indeed, the Hilbertian interpolation argument involved in the bootstrapping does not work when the resolvent of the problem considered is not linear (e.g., Navier-Stokes or unilateral contact problem). Besides, it is well known that such a process should be handled with great care for null-spaces of some partial differential operators, defined on nonsmooth domains (see [13]). A part of this work (mainly Appendix A) is dedicated to an alternative and direct proof of the results of [12] that avoids the bootstrapping and reduces the loss to $\sqrt{\log p}$. The error estimate of the best approximation by mortared *hp*-finite element functions proven here is very useful when applying the *hp*-mortar method to more complicated problems such as the Stokes, the mixed elasticity, or the fluid-structure interaction models.

The extension of the mortar element method to the incompressible Stokes equations (which covers the mixed elasticity problem as well) is realized in [14] for *h*-finite elements and in [15] for spectral elements. After setting a variational mixed formulation of the system written in the velocity-pressure primitive variables, the approximated spaces are constructed using local div-stable mixed finite or spectral elements. The computed velocity satisfies some "mortar" matching conditions on the interfaces while the pressure is free of any constraints. In both papers quoted above, an inf-sup condition is proven with a constant independent of the discretization parameter: h the mesh size for the h-finite elements and N the degree of the local polynomials for the spectral elements. This made it possible to exhibit error estimates with the expected behavior with respect to h or N, which was the main aim of those first pieces of work. Now, when using some preconditioned iterative algorithms such as the conjugate gradient Uzawa procedure with the mass matrix preconditioner (see [16]) or the preconditioned conjugate residual method with a block diagonal preconditioner (see [1,17,18]), an important question comes to mind. *Does the* inf-sup *constant—or equivalently the condition number of the solver—increase with the total number of the subdomains?* The present contribution provides a negative answer. This has an important impact on the rate of convergence of those algorithms, especially when a large number of subdomains are used.

The outline of the paper is as follows. The next section is a discussion of the mixed variational formulation of the nearly incompressible elasticity problem and of the Stokes problem. After the description of the domain decomposition features in Section 2, the mortared space where the velocity is computed so that the pressure space can be constructed. They are based on local div-stable *hp*-finite elements. In view of the numerical analysis of the method, we provide some important approximation results. In particular, a "quasioptimal" convergence rate is given in Proposition 3.1 of the best approximation error of regular velocities by mortared functions. The proof requires technical estimates which are detailed in Appendix A. The setting of the discrete saddle-point problem closes this section. Section 3 is devoted to the study of an inf-sup condition between the mortared velocity space and a reduced pressure space, with a constant that is independent of the discretization parameters (h, p) and of the total number of the subdomains. Such a fundamental result allows us, thanks to the Boland-Nicolaides argument, to derive an optimal inf-sup condition linking the discrete velocity and pressure space which is addressed in Section 4. The last section shows the existence and uniqueness of the approximated solution and provides the optimal convergence rate towards the exact solution.

Notations

Let a Lipschitz domain $\mathcal{C} \subset \mathbb{R}^2$ be given and the generic point of $\mathcal{C}$ is denoted $\boldsymbol{x}$. The classical Lebesgue space of square integrable functions $L^2(\mathcal{C})$ is endowed with the inner product

$$(\phi, \psi)_{L^2(\mathcal{C})} = \int_{\mathcal{C}} \phi\psi \, d\boldsymbol{x},$$

and $L_0^2(\mathcal{C})$ is the subspace of $L^2(\mathcal{C})$ involving the functions of zero average. We use the standard Sobolev space notations, $H^m(\mathcal{C})$, $m \geq 1$, provided with the norm

$$\|\psi\|_{H^m(\mathcal{C})} = \left(\sum_{0 \leq |\alpha| \leq m} \|\partial^\alpha \psi\|^2_{L^2(\mathcal{C})} \right)^{1/2},$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi-index in $\mathbb{N}^2$ and the symbol ∂^α represents a partial derivative. The fractional Sobolev space $H^\tau(\mathcal{C})$, $\tau \in \mathbb{R}_+ \setminus \mathbb{N}$, is defined by its norm (see [19,20])

$$\|\psi\|_{H^\tau(\mathcal{C})} = \left(\|\psi\|^2_{H^m(\mathcal{C})} + \sum_{|\alpha| = m} \int_{\mathcal{C}} \int_{\mathcal{C}} \frac{(\partial^\alpha \psi(\boldsymbol{x}) - \partial^\alpha \psi(\boldsymbol{y}))^2}{|\boldsymbol{x} - \boldsymbol{y}|^{2+2\theta}} \, d\boldsymbol{x} \, d\boldsymbol{y} \right)^{1/2},$$

where $\tau = m + \theta$, m, and $\theta \in]0, 1[$ being the integer part and the fractional part of τ, respectively. The closure in $H^\tau(\mathcal{C})$ of the set $\mathcal{D}(\mathcal{C})$ of indefinitely differentiable functions whose support is contained in $\mathcal{C}$ is denoted $H_0^\tau(\mathcal{C})$.

For any portion of the boundary $\gamma \subset \partial\mathcal{C}$, the space $H^{1/2}(\gamma)$ is the set of the traces over γ of all the functions of $H^1(\mathcal{C})$ and $H^{-1/2}(\gamma)$ is its topological dual space. The duality pairing between $H^{-1/2}(\gamma)$ and $H^{1/2}(\gamma)$ is $\langle ., . \rangle_{*,\gamma}$. The special space $H_{00}^{1/2}(\gamma)$ is the subspace of $H^{1/2}(\gamma)$ of the traces of all functions belonging to $H_0^1(\mathcal{C}, \gamma^c) = \{\psi \in H^1(\mathcal{C}),\ \psi_{|\gamma^c} = 0\}$, where $\gamma^c = \partial\mathcal{C} \setminus \gamma$. It is endowed with the quotient norm

$$\|\phi\|_{H_{00}^{1/2}(\gamma)} = \inf_{\psi \in H_0^1(\mathcal{C}, \gamma^c)} \|\psi\|_{H^1(\mathcal{C})}.$$

For any functional space $X(\mathcal{C})$, the bold symbol $\boldsymbol{X}(\mathcal{C})$ stands for the product $X(\mathcal{C}) \times X(\mathcal{C})$ so that, for instance, $\boldsymbol{H}^\tau(\mathcal{C}) = H^\tau(\mathcal{C}) \times H^\tau(\mathcal{C})$ or $\boldsymbol{H}_{00}^{1/2}(\gamma) = H_{00}^{1/2}(\gamma) \times H_{00}^{1/2}(\gamma)$. The natural norm of $\boldsymbol{X}(\mathcal{C})$ is denoted by $\|.\|_{\boldsymbol{X}(\mathcal{C})}$ and in the case where $\boldsymbol{X}(\mathcal{C})$ is a Hilbert space, the inner product is $(., .)_{\boldsymbol{X}(\mathcal{C})}$.

2. ALMOST INCOMPRESSIBLE MIXED ELASTICITY PROBLEM

The problem of linear elasticity we shall consider consists of determining the displacement $\boldsymbol{u}$, supposed to be small, of a given isotropic and homogeneous elastic material occupying the initial configuration Ω whose boundary is denoted Γ. The solid Ω is subjected to an external force $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and is assumed to be fixed along the whole boundary, which may be viewed as the "worst" case with respect to the analysis we carry out. Indeed, regarding the inf-sup condition we are going to deal with, on both continuous and discrete levels, it is well known that Dirichlet type conditions give rise to more technicalities than any other type of classical boundary conditions (Neumann, Robin, ..., etc.). The mathematical setting of the plain strain model is as follows (see [21,22]): *find* $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ *such that*

$$2\mu(\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_{L^2(\Omega)^4} + \lambda(\operatorname{div} \boldsymbol{u},\ \operatorname{div} \boldsymbol{v})_{L^2(\Omega)} = (\boldsymbol{f}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)}, \qquad \forall v \in \boldsymbol{H}_0^1(\Omega), \tag{2.1}$$

where $\varepsilon(\boldsymbol{v}) = (1/2)\,(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T)$ is the linearized strain tensor and (λ, μ) are the Lamé coefficients. It is known for practitioners that near the incompressibility limit $\lambda \to +\infty$ (or equivalently, when the Poisson modulus $\nu = \lambda/2(\lambda + \mu)$ approaches $1/2$), the finite-element discretization of the variational problem (2.1) suffers from the "numerical locking" phenomenon (see [23]). This is particularly observed as lower degree finite elements are used which is the case not only for the h-version but also for the hp-version, in particular near the corners. Indeed, the hp-finite elements are efficiently used (to achieve exponential convergence of the method) when at the immediate vicinity of the corner a fine "geometrical" mesh is combined with low degree finite elements—h and p are small—(see [5–7,24]) which may generate a numerical locking. The reason why such a phenomenon occurs is the inability of the discrete space to accurately approximate $\boldsymbol{u}$ while satisfying the incompressibility ($\operatorname{div} \boldsymbol{u} = 0$). Several strategies have been designed to overcome locking (see [21]), among which the most popular is the mixed formulation of problem (2.2),(2.3). This allows us to reduce the severity of the constraint ($\operatorname{div} \boldsymbol{u} = 0$) by enforcing it only weakly. Setting ($p = -(\lambda \operatorname{div} \boldsymbol{u}) \in L_0^2(\Omega)$) called the "pressure" and considering it as an independent unknown, we obtain the Hermann variational system: *find* $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ *and* $p \in L_0^2(\Omega)$ *such that*

$$2\mu(\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_{L^2(\Omega)^4} + b(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)}, \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \tag{2.2}$$

$$b(\boldsymbol{u}, q) - \frac{1}{\lambda}\,(p, q)_{L^2(\Omega)} = 0, \qquad \forall q \in L_0^2(\Omega). \tag{2.3}$$

The bilinear form $b(.,.)$ defined over $\boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$ is given by

$$b(\boldsymbol{v}, q) = -(\operatorname{div} \boldsymbol{v}, q)_{L^2(\Omega)}. \tag{2.4}$$

By means of Korn's inequality, the bilinear form $(\varepsilon(.), \varepsilon(.))_{L^2(\Omega)^4}$ is $\boldsymbol{H}_0^1(\Omega)$-elliptic; in addition, $b(.,.)$ satisfies an inf-sup condition with a positive constant α (see [25])

$$\inf_{q \in L_0^2(\Omega)} \sup_{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \|q\|_{L^2(\Omega)}} \geq \alpha. \tag{2.5}$$

Using the Brezzi's saddle-point theory (see [21,25]), problem (2.2),(2.3) is well posed and has a unique solution $(\boldsymbol{u}, p) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$. Moreover, the following stability condition holds:

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \left(\alpha + \frac{1}{\lambda}\right) \|p\|_{L^2(\Omega)} \leq C \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}.$$

REMARK 2.1. INCOMPRESSIBLE STOKES PROBLEM. In the limit situation $\lambda = +\infty$, the term $(1/\lambda)\,(p, q)_{L^2(\Omega)}$ is canceled and problem (2.2),(2.3) boils down to the Stokes equations modeling an incompressible fluid flow, where μ is the viscosity, $\boldsymbol{u}$ the velocity, and p the pressure of the fluid: *find* $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ *and* $p \in L_0^2(\Omega)$ *such that*

$$2\mu(\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_{L^2(\Omega)^4} + b(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v})_{L^2(\Omega)}, \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \tag{2.6}$$

$$b(\boldsymbol{u}, q) = 0, \qquad \forall q \in L_0^2(\Omega). \tag{2.7}$$

The mortar finite-element discretization proposed and analyzed here is still valid for this case and automatically provides similar convergence rates. So the results proven hereafter still hold for the Stokes problem. In addition, problem (2.2),(2.3) with large value of λ may be viewed as a stabilization process of some unstable approximation of (2.6),(2.7) and may ensure fast convergence of some iterative method (see [26]).

3. THE MORTAR *hp*-FINITE-ELEMENT METHOD

This section is a description of the approximation of problem (2.2),(2.3) by *hp*-mixed finite elements of degree $r \in \mathbb{N}$ (taken instead of p to avoid confusion with the pressure). To omit heavy technicalities which are not necessary for our purpose so as to focus on the div-stability in the nonconforming domain decomposition context (which is the core of the paper), we shall consider only *hp*-finite elements that have proved to be uniformly stable in the conforming case.

The framework of the mortar element method (see [8]) proceeds by breaking up the domain Ω into k^* nonoverlapping subdomains that are assumed polygonally shaped for simplicity. We examine only *conforming* decompositions, that is, when they are considered as macromeshes and the subdomains as macroelements such that the intersection of two closed subdomains $\bar{\Omega}_k \cap \bar{\Omega}_\ell$ is either empty, reduced to a common vertex, or to a common edge. If Ω_k and Ω_ℓ are adjacent along a common edge, it is denoted $\Gamma_{k\ell}$, its extreme points are $\{c_{k\ell}^1, c_{k\ell}^2\}$, and $\boldsymbol{n}_{k\ell}$ is the unit normal vector oriented from Ω_k towards Ω_ℓ so that $\boldsymbol{n}_{\ell k} = -\boldsymbol{n}_{k\ell}$. Clearly, $\Gamma_{k\ell}$ is meaningless when Ω_k and Ω_ℓ do not share any common edge, and for convenience $\underline{k}$ is the set of indices $\ell \neq k$ so that $\Gamma_{k\ell}$ exists. Additionally, it is current to assume that the portion of $\partial\Omega_k$ contained in $\partial\Omega$ is supposed to be a union of complete edges. When needed, $\boldsymbol{n}_k$ specifies the outward normal on the whole boundary $\partial\Omega_k$. We will denote both $\Gamma_{k\ell}$ and $\Gamma_{\ell k}$ by $\Gamma_{k\ell}$ where $k < \ell$.

The mortar approximation of the mixed elasticity and/or Stokes problem is based on a local use of, e.g., the finite-element $\mathbb{P}_r/\mathcal{P}_{r-1}$ which is uniformly div-stable (see [27]). Possible alternatives consist of taking the uniformly div-stable Bernardi-Maday mixed element $\mathbb{P}_r/\mathbb{P}_{[\tau r]}$ with $\tau < 1$, or again the (nonuniformly) div-stable Taylor-Hood mixed element $\mathcal{P}_r/\mathcal{P}_{r-1}$ (see [28]) or for the h-version, Bercovier-Pironneau $\mathcal{P}_1\text{iso}\mathcal{P}_2/\mathcal{P}_1$ element, Crouzeix-Raviart or Arnold-Brezzi-Fortin bubble finite elements (see [21,25]). The discretization used in the subdomain Ω_k is specified by the parameter $\delta_k = (h_k, r_k)$ where h_k is supposed to decrease to zero and r_k to increase to infinity and we set $\delta = (\delta_1, \dots, \delta_{k^*})$. For any k $(1 \le k \le k^*)$, we define a quadrangular partition $\mathcal{T}_k^\delta$ of Ω_k. The maximum size of this triangulation is h_k. This mesh is assumed to be regular in the classical sense [22] and let us denote, for any $\kappa \in \mathcal{T}_k^\delta$, F_κ the invertible affine transformation mapping the reference square $\hat{\kappa} = [0,1]^2$ into κ. Notice that $(\mathcal{T}_k^\delta)_k$ are generated independently and there is no reason why two meshes of neighbor subdomains should coincide at the interface. For any $\kappa \in \mathcal{T}_k^\delta$ and any $r \in \mathbb{N}$, $\mathbb{P}_r(\kappa)$ stands for the set of polynomials of degree $\le r$ in each space direction while $\mathcal{P}_r(\kappa)$ is the set of polynomials of total degree $\le r$. The local velocity spaces are then chosen to be

$$\boldsymbol{X}^\delta(\Omega_k) = \left\{\boldsymbol{v}_k^\delta \in \mathcal{C}\left(\bar{\Omega}_k\right)^2,\ \forall \kappa \in \mathcal{T}_k^\delta,\ \boldsymbol{v}_{k|\kappa}^\delta \circ (F_\kappa)^{-1} \in \mathbb{P}_{r_k}\left(\hat{\kappa}\right)^2,\ \boldsymbol{v}_{k|\partial\Omega_k \cap \partial\Omega}^\delta = 0\right\}.$$

The local discrete pressure space is defined as follows:

$$Q^\delta(\Omega_k) = \left\{q_k^\delta \in \mathcal{C}\left(\bar{\Omega}_k\right),\ \forall \kappa \in \mathcal{T}_k^\delta,\ q_{k|\kappa}^\delta \in \mathcal{P}_{r_k - 1}(\kappa)\right\}.$$

Given these tools, the approximation $(\boldsymbol{v}^\delta, q^\delta)$ is taken locally in $\boldsymbol{X}^\delta(\Omega_k) \times Q^\delta(\Omega_k)$ so that $\boldsymbol{v}^\delta = (\boldsymbol{v}_k^\delta)_{1\le k\le k^*}$ are glued together across the interfaces by some suitable matching conditions. Making these conditions explicit requires the use of a finite-element space built on the interfaces. Each $\Gamma_{k\ell}$ inherits a one-dimensional partition $t_{k\ell}^\delta$ from either $\mathcal{T}_k^\delta$ or $\mathcal{T}_\ell^\delta$, let us say from $\mathcal{T}_\ell^\delta$ for $k < \ell$ (the trace of $\mathcal{T}_k^\delta$ on $\Gamma_{k\ell}$ is denoted $t_{\ell k}^\delta$). It is assumed to be an (M)-mesh in the sense of Crouzeix-Thomée (see [29]).

Let $t^\delta = \bigcup_{i=0}^{i^*} t_i$ be a given 1D-mesh, where t_i and t_{i+1} are neighbors for any i $(0 \le i \le i^*-1)$. Setting $h = \sup_{0\le i\le i^*} |t_i|$, t^δ is said to be an (M)*-mesh if*

$$\frac{|t_i|}{|t_j|} \le C\beta^{|i-j|}, \qquad \forall i,j\,(0 \le i,j \le i^*), \tag{3.1}$$

where $1 \le \beta < (r+1)^2$ and C does not depend on h.

Then, for the enforcement of the nonconforming matching conditions, we need to introduce the space of Lagrange multipliers on each $\Gamma_{k\ell}$

$$M^\delta(\Gamma_{k\ell}) = \left\{\psi^\delta \in \mathcal{C}\left(\bar\Gamma_{k\ell}\right),\ \forall t \in t_{k\ell}^\delta,\ \psi_{|t}^\delta \in \mathcal{P}_{r_\ell}(t),\ \psi_{|t}^\delta \in \mathcal{P}_{r_\ell-1}(t) \text{ if } \boldsymbol{c}_{k\ell}^1 \in t \text{ or } \boldsymbol{c}_{k\ell}^2 \in t\right\},$$

and we set $\boldsymbol{M}^\delta(\Gamma_{k\ell}) = M^\delta(\Gamma_{k\ell}) \times M^\delta(\Gamma_{k\ell})$. By a duality argument, it can be proven that the following estimate holds: for any $\nu \ge 1/2$, there exists a constant C such that: $\forall \Psi \in \boldsymbol{H}^{\nu-1/2}(\Gamma_{k\ell})$,

$$\inf_{\Psi^\delta \in \boldsymbol{M}^\delta(\Gamma_{k\ell})} \left\|\Psi - \Psi^\delta\right\|_{\boldsymbol{H}^{-1/2}(\Gamma_{k\ell})} \le C\frac{h_k^\eta}{r_k^\nu}\|\Psi\|_{\boldsymbol{H}^{\nu-1/2}(\Gamma_{k\ell})} \tag{3.2}$$

with $\eta = \min(\nu, r)$. The global nonconforming velocity approximation space is then given by

$$\boldsymbol{X}^\delta(\Omega) = \Bigg\{\boldsymbol{v}^\delta = \left(\boldsymbol{v}_k^\delta\right)_k \in \boldsymbol{L}^2(\Omega), \text{ such that } \boldsymbol{v}_k^\delta \in \boldsymbol{X}^\delta(\Omega_k) \text{ and} : \forall k,\ell$$
$$\forall \Psi_{k\ell}^\delta \in \boldsymbol{M}^\delta(\Gamma_{k\ell}),\ \int_{\Gamma_{k\ell}} \left(\boldsymbol{v}_k^\delta - \boldsymbol{v}_\ell^\delta\right).\Psi_{k\ell}^\delta\, d\Gamma = 0\Bigg\}.$$

The discrete space for the pressure is defined to be

$$Q^\delta(\Omega) = \left\{q^\delta = \left(q_k^\delta\right)_k \in L_0^2(\Omega),\ q_k^\delta \in Q^\delta(\Omega_k)\right\}.$$

The space $Q^\delta(\Omega)$ is provided with the $L^2(\Omega)$-norm while the space $\boldsymbol{X}^\delta(\Omega)$, not being embedded in $\boldsymbol{H}_0^1(\Omega)$, is then endowed with the Hilbertian broken norm

$$\|\boldsymbol{v}^\delta\|_* = \left(\sum_{k=1}^{k^*} \|\boldsymbol{v}_k^\delta\|_{\boldsymbol{H}^1(\Omega_k)}^2\right)^{1/2}.$$

REMARK 3.1. Let us stress the fact that only the velocity space is mortared while the pressure is not subjected to any particular constraints across the interfaces. The family $(\Gamma_{k\ell})_{k<\ell}$ equipped with the meshes $(t_{k\ell}^\delta)_{k<\ell}$ are the mortars; they are nonmortars when endowed with $(t_{\ell k}^\delta)_{k<\ell}$, and the functions $\Phi^\delta = (\boldsymbol{v}_{k|\Gamma_{k\ell}}^\delta)_{k<\ell}$ are the mortar functions.

The space $Q^\delta(\Omega)$ satisfies the following approximation result [2]: $\forall q \in L_0^2(\Omega)$ with $q_k = q_{|\Omega_k} \in H^{\nu_k}(\Omega_k)$,

$$\inf_{q^\delta \in Q^\delta(\Omega)} \left\|q - q^\delta\right\|_{L^2(\Omega)} \le C\sum_{k=1}^{k^*} \frac{h_k^{\eta_k}}{r_k^{\nu_k}}\|q_k\|_{H^{\nu_k}(\Omega_k)}, \tag{3.3}$$

with $\eta_k = \min(\nu_k, r_k)$. So far, when the meshes $(t_{k\ell}^\delta)_{k<\ell}$ satisfy the (M)-mesh criteria (3.1), the only known approximation rate by the functions of $\boldsymbol{X}^\delta(\Omega)$ is given in [9] and it is suboptimal (actually polluted by a $r_k^{3/4}$-factor). At the expense of a technical work, it is possible to recover the "quasioptimality" where the pollution term can be reduced to a $|\log r_k|^{1/2}$.

PROPOSITION 3.1. *Assume that* $v \in H_0^1(\Omega)$ *with* $v_k = v_{|\Omega_k} \in H^{\nu_k+1}(\Omega_k)$, $\nu_k > 1/2$,

$$\inf_{v^\delta \in X^\delta(\Omega)} \left\| v - v^\delta \right\|_* \leq C \sum_{k=1}^{k^*} \frac{h_k^{\eta_k}}{r_k^{\nu_k}} \left| \log r_k \right|^{1/2} \| v_k \|_{H^{\nu_k+1}(\Omega_k)}, \tag{3.4}$$

with $\eta_k = \min(\nu_k, r_k)$.

PROOF. As noticed in [9] for the *hp*-version of the mortar finite-element method, and earlier in [8] for the *h*-version, this estimate is primarily dependent on the approximation quality of the mortar projection $\pi_{k\ell}^\delta$ ranging $H^1(\Gamma_{k\ell})$ on the space

$$W^\delta(\Gamma_{k\ell}) = \left\{ \chi_{k\ell}^\delta \in \mathcal{C}\left(\bar{\Gamma}_{k\ell}\right),\ \forall t \in t_{k\ell}^\delta,\ \chi_{k\ell|t}^\delta \in \mathcal{P}_{r_\ell}(t) \right\} = X^\delta(\Omega_k)_{|\Gamma_{k\ell}}$$

and defined as follows: $\forall \chi \in H^1(\Gamma_{k\ell})$,

$$\begin{aligned} \pi_{k\ell}^\delta \chi(c) &= \chi(c), && \forall c \in \left\{ c_{k\ell}^1, c_{k\ell}^2 \right\}, \\ \int_{\Gamma_{k\ell}} \left(\chi - \pi_{k\ell}^\delta \chi \right) \psi_{k\ell}^\delta \, d\Gamma &= 0, && \forall \psi_{k\ell}^\delta \in M^\delta(\Gamma_{k\ell}). \end{aligned} \tag{3.5}$$

We need in particular to evaluate the error $\|\chi - \pi_{k\ell}^\delta \chi\|_{H_{00}^{1/2}(\Gamma_{k\ell})}$ for $\chi \in H^{\nu_k+1/2}(\Gamma_{k\ell})$ because, following [8] (see also [12, Proof of Lemma 4.3]), the convergence rate of $\inf_{v^\delta \in X^\delta(\Omega)} \|v - v^\delta\|_*$ is exactly of the same order. As the proof of such a result requires a lot of technical manipulations, it is postponed to Appendix A, Theorem A.2.1. ∎

REMARK 3.2. Recall that the local spaces $Q^\delta(\Omega_k) \cap L_0^2(\Omega)$ and $X^\delta(\Omega_k) \cap H_0^1(\Omega)$ satisfy an inf-sup condition with a constant α_k that is independent of $\delta_k = (h_k, r_k)$ (see [27]). This means that for any $q_k^\delta \in Q^\delta(\Omega_k)$, there exists $v_k^\delta \in X^\delta(\Omega_k) \cap H_0^1(\Omega_k)$ such that

$$\begin{aligned} -\left(\operatorname{div} v_k^\delta, q_k^\delta\right)_{L^2(\Omega_k)} &= \left\| q_k^\delta \right\|_{L^2(\Omega_k)}^2, \\ \alpha_k \left\| v_k^\delta \right\|_{H^1(\Omega_k)} &\leq \left\| q_k^\delta \right\|_{L^2(\Omega_k)}. \end{aligned} \tag{3.6}$$

We are now in position to investigate the discrete version of the mixed problem (2.2),(2.3) which is written as: *find* $(u^\delta, p^\delta) \in X^\delta(\Omega) \times Q^\delta(\Omega)$ *satisfying*

$$a\left(u^\delta, v^\delta\right) + b(v^\delta, p^\delta) = \left(f, v^\delta\right)_{L^2(\Omega)}, \qquad \forall v^\delta \in X^\delta(\Omega), \tag{3.7}$$

$$b\left(u^\delta, q^\delta\right) - \frac{1}{\lambda}\left(p^\delta, q^\delta\right)_{L^2(\Omega)} = 0, \qquad \forall q^\delta \in Q^\delta(\Omega). \tag{3.8}$$

In (3.7),(3.8), we have set: $\forall u^\delta, v^\delta \in X^\delta(\Omega)$, $\forall q^\delta \in Q^\delta(\Omega)$,

$$a\left(u^\delta, v^\delta\right) = 2\mu \sum_{k=1}^{k^*} \left(\varepsilon\left(u_k^\delta\right), \varepsilon\left(v_k^\delta\right)\right)_{L^2(\Omega_k)^4} \quad \text{and} \quad b\left(v^\delta, q^\delta\right) = -\sum_{k=1}^{k^*} \left(\operatorname{div} v_k^\delta, q_k^\delta\right)_{L^2(\Omega_k)}.$$

Actually $b(.,.)$ is an extension of the bilinear form (2.4) and is still denoted $b(.,.)$. The existence and uniqueness result for this problem depends on the bilinear forms $a(.,.)$ and $b(.,.)$. They are both trivially continuous. Adapting the proof of [8] of the Poincaré-Friedrichs inequality and using the Korn's inequality, it is not hard to show that $a(.,.)$ is elliptic on $X^\delta(\Omega)$ with a constant uniformly bounded from below (see [8]). The principal remaining point is the verification of a uniform inf-sup condition on $b(.,.)$. We resort to Boland and Nicolaides' argument [30] which reduces the problem to an evaluation of local inf-sup conditions which are readily checked in (3.6) and to the proof of a global inf-sup condition on a smaller discrete pressure space. This issue is addressed in the following section.

REMARK 3.3. In practice, the bilinear forms, $a(.,.)$, $b(.,.)$, and $(.,.)_{L^2(\Omega)}$, involved in the discrete equations are evaluated using some suitable numerical integration methods such as, e.g., the Gauss-Lobatto-Legendre quadrature rule. Trivially, doing so introduces a new error source. This contribution to the error may be taken into account using classical techniques (see [31]) which is not that hard (that is why we prefer to avoid such a concern). However, the most important point to be retained is that the final asymptotic convergence rate is preserved.

4. OPTIMAL inf-sup CONDITION ON A REDUCED PRESSURE SPACE

It is stated in [14, Theorem 4.2] that an inf-sup condition links the space $X^\delta(\Omega)$ and the finite-dimensional space of piecewise constant pressures

$$\check{Q}(\Omega) = \left\{ \check{q} = (\check{q}_k) \in \mathbb{R}^{k^*},\ (\check{q}, 1)_{L^2(\Omega)} = \sum_{k=1}^{k^*} \check{q}_k |\Omega_k| = 0 \right\},$$

with a constant that does not depend on the parameter δ. The only question that remains regards the behavior of that constant versus the total number of the subdomains. Such a result is very useful in the analysis of some iterative preconditioned algorithms when applied to solve the discrete problem (3.7),(3.8) (see [1,17,18]). The subject of the sequel is to discuss an alternative and shorter proof which allows us to show that the inf-sup constant could be chosen so that it does not increase with the total number k^* of the subdomains.

PROPOSITION. *There exists a constant $\check{\alpha}$ independent of the discretization parameter δ as well as the total number k^* of subdomains so that the following inf-sup condition holds:*

$$\inf_{\check{q} \in \check{Q}(\Omega)} \sup_{v^\delta \in X^\delta(\Omega)} \frac{b\left(v^\delta, \check{q}^\delta\right)}{\|v^\delta\|_* \, \|\check{q}^\delta\|_{L^2(\Omega)}} \geq \check{\alpha}. \tag{4.1}$$

PROOF. Since the inf-sup condition (2.5) holds in the continuous case, an equivalent statement of the proposition is that for any $v \in H_0^1(\Omega)$, there exists $v^\delta \in X^\delta(\Omega)$ such that

$$b\left(v - v^\delta, \check{q}^\delta\right) = 0, \qquad \forall \check{q}^\delta \in \check{Q}(\Omega), \tag{4.2}$$

$$\gamma \left\|v^\delta\right\|_* \leq \|v\|_{H^1(\Omega)}, \tag{4.3}$$

where $\gamma > 0$ depends neither on δ nor on k^*. Then the constant $\check{\alpha}$ could be taken equal to the product $\alpha\gamma$. Statements (4.2),(4.3) are established in three steps.

(i) For any interface $\Gamma_{k\ell}$, let us choose $\Phi_{k\ell}^\delta \in \mathcal{P}_2^0(\Gamma_{k\ell})^2$ (the set of second degree polynomials that vanish at the end points of $\Gamma_{k\ell}$) satisfying

$$\Phi_{k\ell}^\delta . t_{k\ell} = 0 \qquad \text{and} \qquad \int_{\Gamma_{k\ell}} \Phi_{k\ell}^\delta . n_{k\ell} \, d\Gamma = 1,$$

where $\|\Phi_{k\ell}^\delta\|_{H_{00}^{1/2}(\Gamma_{k\ell})}$ is only dependent on the length of the edge $\Gamma_{k\ell}$.

(ii) Let v be given in $H_0^1(\Omega)$, and define $\Phi^\delta \in \mathcal{C}(\mathcal{S})^2$ such that: $\forall\, k\ell$,

$$\Phi_{|\Gamma_{k\ell}}^\delta = \left(\int_{\Gamma_{k\ell}} v.n \, d\Gamma \right) \Phi_{k\ell}^\delta, \qquad \text{so that } \int_{\Gamma_{k\ell}} \left(v - \Phi_{|\Gamma_{k\ell}}^\delta \right) . n_{k\ell} \, d\Gamma = 0.$$

It is clear that $\Phi_{|\partial\Omega_k}^\delta \in H^{1/2}(\partial\Omega_k)$ and

$$\left\| \Phi_{|\partial\Omega_k}^\delta \right\|_{H^{1/2}(\partial\Omega_k)} \leq c_k \sum_{\ell \in \underline{k}} \left| \int_{\Gamma_{k\ell}} v.n \, d\Gamma \right| \leq C_k \|v\|_{H^1(\Omega_k)}.$$

The constants c_k and C_k depend only on the size of Ω_k. Then, using a stable extension of the trace function $\Phi_{|\partial\Omega_k}^\delta$ we construct $v_k^\delta \in X^\delta(\Omega_k)$ of the minimal degree ($= 2$) whose trace on $\partial\Omega_k$ coincides with $\Phi_{|\partial\Omega_k}^\delta$ and satisfies the estimate (see [32, Theorem 5.1])

$$\left\|v^\delta\right\|_{H^1(\Omega_k)} \leq c_k' \left\| \Phi_{|\partial\Omega_k}^\delta \right\|_{H^{1/2}(\partial\Omega_k)} \leq C_k' \|v\|_{H^1(\Omega_k)}. \tag{4.4}$$

(iii) Setting $\boldsymbol{v}^\delta = (\boldsymbol{v}_k^\delta)_{1\le k\le k^*}$, it is straightforward that $\boldsymbol{v}^\delta$ is continuous on the whole domain Ω and then belongs to $\boldsymbol{X}^\delta(\Omega)$. Because of (4.4), it is seen directly that (4.2) is valid with $\gamma = (\max_k C_k')^{-1}$ which is dependent only on the shape of the subdomains and not their total number. Besides, by Green's formula, we have: $\forall\, \check{q}^\delta \in \check{Q}(\Omega)$,

$$b\left(\boldsymbol{v}-\boldsymbol{v}^\delta, \check{q}^\delta\right) = \sum_{k<\ell} \left[\check{q}^\delta\right] \int_{\Gamma_{k\ell}} \left(\boldsymbol{v}-\boldsymbol{v}^\delta\right).\boldsymbol{n}\, d\Gamma.$$

The symbol [.] stands for the jump function. Since $\boldsymbol{v}^\delta_{|\Gamma_{k\ell}} = \Phi^\delta_{|\Gamma_{k\ell}}$, it becomes clear that: $\forall\, k\ell$,

$$\int_{\Gamma_{k\ell}} \left(\boldsymbol{v}-\boldsymbol{v}^\delta\right).\boldsymbol{n}\, d\Gamma = 0,$$

which completes the proof of (4.3). ∎

REMARK 4.1. The result of the proposition is readily extended to arbitrary domain decomposition using the same technical tools combined with those already developed in [15]. Since such a proof is very technical but does not require any new tools, we have chosen to consider only the simple case.

5. BOLAND-NICOLAIDES ARGUMENT AND ERROR ESTIMATE

Back to the global pressure space $Q^\delta(\Omega)$, we intend to prove by the Boland-Nicolaides method an optimal mortar inf-sup condition between $\boldsymbol{X}^\delta(\Omega)$ and $Q^\delta(\Omega)$.

PROPOSITION 5.1. *There exists a constant α' that depends only on the shape of the subdomains $(\Omega_k)_{1\le k\le k^*}$ so that the following* inf-sup *condition holds:*

$$\inf_{q^\delta\in Q^\delta(\Omega)} \sup_{\boldsymbol{v}^\delta\in \boldsymbol{X}^\delta(\Omega)} \frac{b\left(\boldsymbol{v}^\delta, q^\delta\right)}{\|\boldsymbol{v}^\delta\|_* \|q^\delta\|_{L^2(\Omega)}} \ge \alpha'. \tag{5.1}$$

PROOF. It is made following the idea of [30]. Letting $q^\delta = (q_k^\delta)_{1\le k\le k^*}$ be in $Q^\delta(\Omega)$, it may be decomposed as: $\forall\, k (1\le k\le k^*)$,

$$q_k^\delta = \tilde{q}_k^\delta + \check{q}_k^\delta, \qquad \text{with } \check{q}_k^\delta = \frac{1}{|\Omega_k|}\int_{\Omega_k} q_k^\delta(\boldsymbol{x})\, d\boldsymbol{x}.$$

Since the function $\tilde{q}_k^\delta \in Q^\delta(\Omega_k)\cap L_0^2(\Omega_k)$, there exists a function $\tilde{\boldsymbol{v}}_k^\delta \in \boldsymbol{X}^\delta(\Omega_k)\cap \boldsymbol{H}_0^1(\Omega_k)$ verifying (3.6). Then, we define the function $\tilde{\boldsymbol{v}}^\delta = (\tilde{\boldsymbol{v}}_k^\delta)_{1\le k\le k^*}$. Furthermore, by the previous proposition, we construct $\check{\boldsymbol{w}}^\delta \in \boldsymbol{X}^\delta(\Omega)$ so that

$$b\left(\check{\boldsymbol{w}}^\delta, \check{q}^\delta\right) = \left\|\check{q}^\delta\right\|^2_{L^2(\Omega)} \qquad \text{and} \qquad \check{\alpha}\left\|\check{\boldsymbol{w}}^\delta\right\|_{H^1(\Omega)} \le \left\|\check{q}^\delta\right\|_{L^2(\Omega)}.$$

Next, we choose $\check{\boldsymbol{v}}^\delta = (\check{\boldsymbol{v}}_k^\delta)_{1\le k\le k^*}$ with $\check{\boldsymbol{v}}_k^\delta = \check{\boldsymbol{w}}_k^\delta + d_k\tilde{\boldsymbol{v}}_k^\delta$. The coefficients $(d_k)_{1\le k\le k^*}$ are computed so that the mixed terms $\{(\operatorname{div}\check{\boldsymbol{v}}^\delta, \tilde{q}^\delta)_{L^2(\Omega_k)},\ 1\le k\le k^*\}$ vanish, and then

$$d_k = -\frac{\left(\operatorname{div}\check{\boldsymbol{w}}_k^\delta, \tilde{q}_k^\delta\right)_{L^2(\Omega_k)}}{\left\|\tilde{q}_k^\delta\right\|^2_{L^2(\Omega_k)}}.$$

Taking $\boldsymbol{v}^\delta = \check{\boldsymbol{v}}^\delta + \tilde{\boldsymbol{v}}^\delta$, it is readily checked that

$$b\left(\boldsymbol{v}^\delta, q_k^\delta\right) = \left\|\tilde{q}_k^\delta\right\|^2_{L^2(\Omega_k)} + \left\|\check{q}_k^\delta\right\|^2_{L^2(\Omega_k)},$$

and $\alpha'\|\boldsymbol{v}^\delta\|_{\boldsymbol{H}^1(\Omega)} \le \|q^\delta\|_{L^2(\Omega)}$ where α' may be chosen so that

$$\alpha' = \frac{1}{2}\min(1,\check{\alpha}) \min_{1\le k\le k^*} \frac{\alpha_k}{\sqrt{1+\alpha_k^2}},$$

which depends only on the size of the subdomains. ∎

The immediate consequence is the well-posedness of the discrete problem. Indeed, by the saddle-point theory, we have the following result.

PROPOSITION 5.2. *The discrete problem (3.7),(3.8) has only one solution* $(\boldsymbol{u}^\delta, p^\delta) \in \boldsymbol{X}^\delta \times Q^\delta(\Omega)$ *which is uniformly stable with respect to the data* $\boldsymbol{f}$,

$$\|\boldsymbol{u}^\delta\|_* + \left(\alpha' + \frac{1}{\lambda}\right) \|p^\delta\|_{L^2(\Omega)^2} \leq C\|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{5.2}$$

In order to perform the analysis and to derive the convergence rate towards the exact solution, we adapt as is done in [33] (see also [14]) the approximation theory of the saddle-point problems to our nonconforming discretization. The consistency error, caused by the nonconformity of the mortar approach, is handled in a standard way using estimate (3.2) (cf. [8,12]). Besides, because the inf-sup condition of Proposition 5.1 is independent of δ, techniques used for standard h-finite-element approximation work as well. Indeed, the error of the best approximation of the velocity $\boldsymbol{u}$ by vector-valued function of

$$\boldsymbol{V}^\delta(\Omega) = \left\{\boldsymbol{v}^\delta \in \boldsymbol{X}^\delta(\Omega),\ b\left(\boldsymbol{v}^\delta, q^\delta\right) = 0,\ \forall q^\delta \in Q^\delta(\Omega)\right\}$$

is equivalent to that provided by the best fit of $\boldsymbol{u}$ by the functions of $\boldsymbol{X}^\delta(\Omega)$. This rate is quasioptimal as given in Proposition 3.1. Moreover, the approximation error is optimal for the pressure (3.3). The final result is given by the following theorem.

THEOREM 5.3. *Assume that the exact solution* $(\boldsymbol{u}, p) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$ *satisfies the regularity assumptions*

$$\boldsymbol{u}_k = \boldsymbol{u}_{|\Omega_k} \in \boldsymbol{H}^{\nu_k+1}(\Omega_k), \qquad p_k = p_{|\Omega_k} \in H^{\nu_k}(\Omega_k), \qquad \forall k\ (1 \leq k \leq k^*).$$

Then the discrete solution satisfies

$$\begin{aligned} \left\|\boldsymbol{u} - \boldsymbol{u}^\delta\right\|_* + \alpha(\lambda)\left\|p - p^\delta\right\|_{L^2(\Omega)} &\leq C \sum_{k=1}^{k^*} \frac{h_k^{\eta_k}}{r_k^{\nu_k}} \\ \times \left(|\log r_k|^{1/2} \|\boldsymbol{u}_k\|_{\boldsymbol{H}^{\nu_k+1}(\Omega_k)} + \alpha(\lambda)\|p_k\|_{H^{\nu_k}(\Omega_k)}\right)&, \end{aligned} \tag{5.3}$$

with $\eta_k = \min(\nu_k, r_k)$ *and where* $\alpha(\lambda)$ *is bounded away from 0 and blows up like* $1/\lambda$ *for* $\lambda \to 0$.

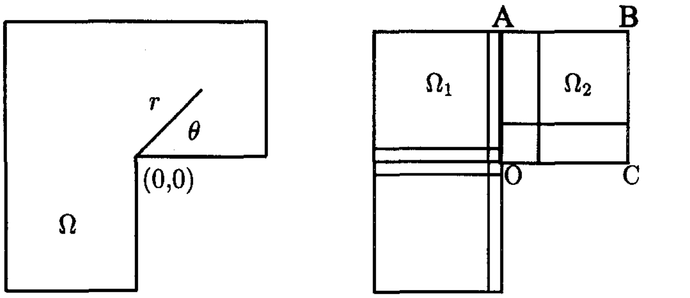
REMARK 5.1. For several mixed hp-finite elements (see [34]), the local inf-sup constant α_k may depend on the parameter r_k, and the constant α' of Proposition 5.1 will have the same behavior with respect to $(r_k)_k$. Moreover, we emphasize the fact that it remains independent of the total number of subdomains because Proposition 4.1 is still valid. In such situations, we can prove that the error of the velocity remains optimal; the approximation error $\inf_{\boldsymbol{v}^\delta \in \boldsymbol{V}^\delta(\Omega)} \|\boldsymbol{u} - \boldsymbol{v}^\delta\|_*$ has to be evaluated directly like in [15]. On the contrary, the accuracy on the pressure deteriorates by a factor $(\min_{1 \leq k \leq k^*} \alpha_k)^{-1}$ and is subject to some numerical locking.

6. NUMERICAL EXPERIMENTS

In this section, we investigate the computational performance of the mortar element method for the Stokes boundary value problem for viscous incompressible fluid flow whose variational form is given by (2.6),(2.7). Actually, for the motion equation (2.6), we have considered the following form:

$$\nu \int_\Omega \nabla \boldsymbol{u}.\nabla \boldsymbol{v}\, d\boldsymbol{x} - \int_\Omega p(\operatorname{div} \boldsymbol{v})\, d\boldsymbol{x} = \int_\Omega \boldsymbol{f}.\boldsymbol{v}\, d\boldsymbol{x}, \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega, \Gamma_D),$$

where $\nu > 0$ is the kinematic viscosity which is related to the Reynolds number of the flow. The right-hand side $\boldsymbol{f}$ is a given body force per unit mass. Γ_D is the Dirichlet part of the boundary.



(a) L-shaped domains. (b) Tensor product mesh for $m = n = 2$.

Figure 1.

We are involved with the case where the viscosity $\nu = 1$. The problem is set on the L-shaped domain Ω (shown in Figure 1a) that is subdivided in two subdomains Ω_1 and Ω_2, by the interface OA). We impose the Dirichlet boundary condition on both velocity components along the edge OC and Neumann boundary condition on the remainder of the boundary. The nonconforming method is implemented as a mixed method. In our experiments, we restrict our meshes to be *tensor product* and *uniform* with Ω_1 divided into $2m^2$ rectangles (with the mesh always being symmetric about $y = 0$) and Ω_2 divided into n^2 rectangles, as in Figure 1b. We consider the following exact solution:

$$\boldsymbol{u}^{\mathsf{T}}(x,y) = \left(-e^x(y\cos(y) + \sin(y))\,, e^x y\sin(y)\right), \qquad p(x,y) = 2e^x\sin(y).$$

It can be easily checked that the corresponding $\boldsymbol{f} = 0$.

REMARK 6.1. It is well known that the domain in Figure 1 will result in a strong singularity which occurs at the reentrant corner O. The exact solution that we test in this paper is analytical in the closure of Ω and one cannot expect solutions to behave so nicely at such reentrant corners. Nevertheless, smooth solutions arise, for example, in smooth domains and it is hence reasonable to investigate the numerical performance of our method for such exact solutions, which is the main focus herein.

We choose the velocity-pressure space combination to be $\mathbb{P}_p/\mathcal{P}_{p-1}$. For both our experiments, we plot the percentage relative energy norm error for the velocity and the L^2 norm error for the pressure, with respect to the total number of degrees of freedom.

Results for h-version

We take m grid points along both the x- and y-axis for Ω_1 (top half) and n for Ω_2, and use the mortar finite-element method with the combinations $(m,n) \in \{(2,3),(4,6),\dots,(10,15)\}$. Note that for each of these combinations, the meshes are incompatible, and hence, are ideal for testing the performance of our method. The approximation order for the velocity is chosen to be quadratic ($p = 2$) and this implies a linear approximation of the pressure. In Figure 2, we clearly observe an optimal $O(h^p)$ rate in both the velocity and the pressure variables.

Results for p-version

Here, we fix the mesh to be $(m,n) = (2,3)$ and observed the convergence rates for increasing polynomial degree from $p = 2,\dots,8$. Figure 3 clearly illustrates the expected p-version convergence and it also indicates that the p-version easily surpasses the h-version in accuracy, due to higher convergence rates.

Comments

The numerical results presented in this paper clearly suggest that the mortar finite-element method is a robust and viable domain decomposition technique for the Stokes problem. Our

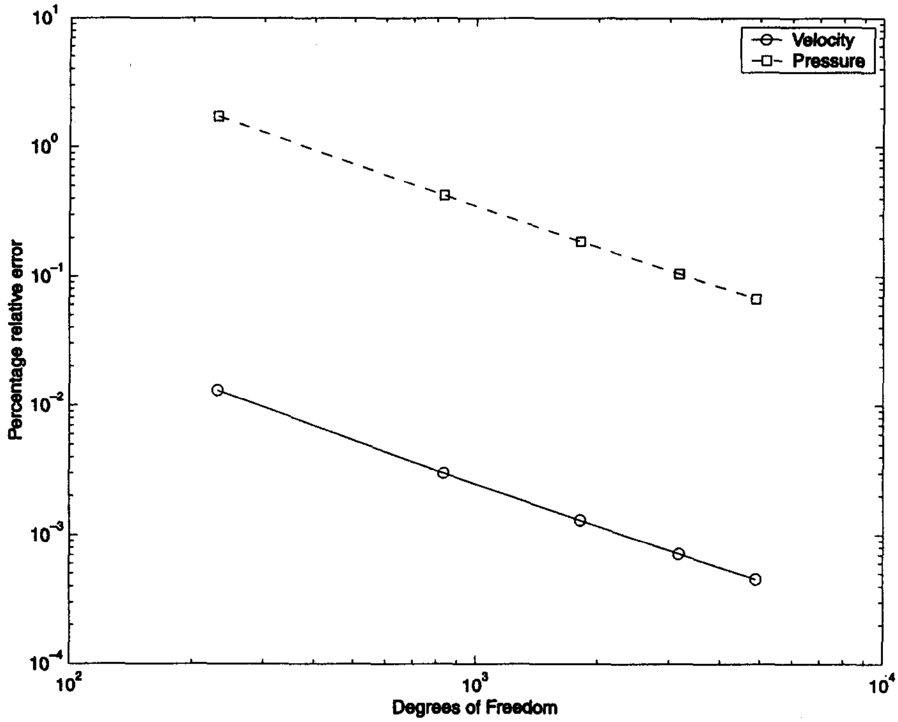


Figure 2. h-version convergence rates for mortar FEM ($p = 2$).

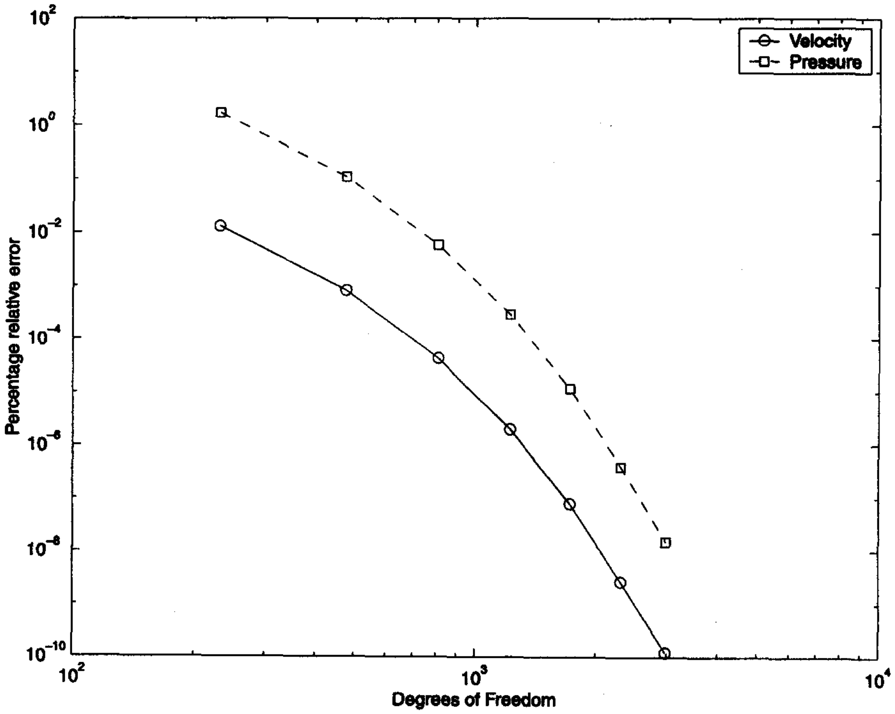


Figure 3. p-version convergence rates for mortar FEM ($(m, n) = (2, 3)$).

results are in correspondence with optimal theoretical results. We expect an exponential convergence for the hp-version in the presence of nonquasiuniform meshes. Also, one can obtain similar results for the mixed elasticity problem. The latter two aspects will be considered in a following paper.

7. CONCLUSION

The results of this study are substantial improvements of those already proven in [14,15]. The statement that the Babuška-Brezzi inf-sup condition for the h-, p-, and hp-mixed mortar finite-element method does not depend on the total number of the subdomains is rigorously established (Propositions 4.1 and 5.1). This additional information is an important concern in view of using, in the mortar context, some iterative substructuring solvers (see [1,17,18]) that have proved to be efficient in the conforming case.

Applying the mortar finite-element method to three dimensions gives rise to severe technical difficulties especially for tetrahedral meshes; we refer to [35] (see also [10]). Recent work based on bubble-stabilization techniques allows a flexible and attractive extension of the h-mortar finite elements to three-dimensional second-order elliptic problems (see [36]). Approximating mixed elasticity and Stokes equations in three dimensions is proposed in [37] and the inf-sup condition is also proven to be optimal in that it does not depend on the parameters discretization nor in the total number of the subdomains. Generalizing these results to the (three-dimensional) hp-mortar finite elements is still an open problem.

The second interesting contribution of this work is specific to hp-mortar approximations. Indeed, the convergence rate of regular functions by hp-mortared finite-element functions is proven to suffer only from $\sqrt{\log p}$ pollution (Proposition 3.1) instead of $p^{3/4}$ proved in [9]. We are confident that such an improvement will be useful in the future and facilitate attempts to widen the list of the applications of the hp-mortar methods.

The combination of both results allows us to derive an error estimate (Theorem 5.3) that is $\sqrt{\log p}$ away from the optimal bound. We note that this estimate is dependent only upon the local features of the exact solution.

APPENDIX A

The aim of this appendix is to prove the "quasioptimal" estimation on the projections $(\pi^\delta_{k\ell})_{k\ell}$ necessary for the evaluation of the best approximation error by the functions of $\boldsymbol{X}^\delta(\Omega)$ given in Proposition 3.1. The proof requires some preliminary results for some polynomial projections. Throughout this appendix and the next one, the symbol $\mathcal{P}^0_p(-1,1)$ stands for the subset of polynomials which vanish at ± 1 and the subspace $\mathcal{P}^R_p(-1,1)$ (respectively, $\mathcal{P}^L_p(-1,1)$) contains all the polynomials of $\mathcal{P}_p(-1,1)$ that vanish at -1 (respectively, 1) so that $\mathcal{P}^R_p(-1,1) \cap \mathcal{P}^L_p(-1,1) = \mathcal{P}^0_p(-1,1)$.

A.1. Analysis of Two L^2-Polynomial Projections on $\mathcal{P}^0_r(-1,1)$

Denote by π_r the orthogonal projection on $\mathcal{P}^0_r(-1,1)$ with respect to the $L^2(-1,1)$-inner product, $\forall \chi \in L^2(-1,1)$, $\pi_r\chi \in \mathcal{P}^0_r(-1,1)$,

$$\int_{-1}^{1} (\chi - \pi_r\chi)\psi_r \, d\xi = 0, \qquad \forall \psi_r \in \mathcal{P}^0_r(-1,1),$$

and let π^*_r stand for the orthogonal projection $\mathcal{P}^0_r(-1,1)$ in $H^1_0(-1,1)$; it may be specified as follows:

$$\int_{-1}^{1} (\chi - \pi^*_r\chi)\, \psi_r \, d\xi = 0, \qquad \forall \psi_r \in \mathcal{P}_{r-2}(-1,1).$$

Recall the optimal approximation results of [31], $\forall \alpha \in [0,1]$, $\forall \nu \geq 0$, $\forall \chi \in H^{1+\nu}(-1,1) \cap H^1_0(-1,1)$,

$$\|\chi - \pi_r\chi\|_{L^2(-1,1)} \leq \frac{C}{r^{1+\nu}} \|\chi\|_{H^{1+\nu}(-1,1)}, \tag{A.1.1}$$

$$\|\chi - \pi^*_r\chi\|_{H^{1/2}_{00}(-1,1)} + \frac{1}{r^{\alpha-1/2}} |\chi - \pi^*_r\chi|_{H^\alpha(-1,1)} \leq \frac{C}{r^{1/2+\nu}} \|\chi\|_{H^{1+\nu}(-1,1)}. \tag{A.1.2}$$

So far, the operator π_r is known to have suboptimal approximation behavior with respect to $H^{1/2}_{00}(-1,1)$-seminorm; i.e., for any $\nu \geq 0$ we have, $\forall\,\chi \in H^{1+\nu}(-1,1) \cap H^1_0(-1,1)$,

$$|\chi - \pi_r\chi|_{H^{1/2}_{00}(-1,1)} \leq C\frac{r^{1/4}}{r^{1/2+\nu}}\,\|\chi\|_{H^{1+\nu}(-1,1)}. \tag{A.1.3}$$

This estimate is $r^{1/4}$ away from optimality; we refer to [38] for the details. The main new point here is that actually the loss is not worse than $\sqrt{\log r}$ and the approximation rate becomes quasioptimal. The proof relies on the bound of $\|L_r\|_{H^{1/2}(-1,1)} \leq C\sqrt{\log r}$, recently provided in [39].

LEMMA A.1.1. *For any $\nu \geq 0$, the following holds: $\forall\,\chi \in H^{1+\nu}(-1,1) \cap H^1_0(-1,1)$,*

$$|\chi - \pi_r\chi|_{H^{1/2}_{00}(-1,1)} \leq C\frac{\sqrt{\log r}}{r^{1/2+\nu}}\,\|\chi\|_{H^{1+\nu}(-1,1)}. \tag{A.1.4}$$

PROOF. It is immediate that

$$\int_{-1}^{1} (\pi_r\chi - \pi_r^*\chi)\,\psi_r\,(1-\xi^2)\;d\xi = 0, \qquad \forall\,\psi_r \in \mathcal{P}_{r-4}(-1,1).$$

Because of the orthogonality of $(L'_r)_r$ with respect to the measure $(1-\xi^2)\,d\xi$, it turns out that $(\pi_r\chi - \pi_r^*\chi)$ must belong to the span of $\{L'_{r+1}, L'_r, L'_{r-1}, L'_{r-2}\}$. Using the fact that $(\pi_r\chi - \pi_r^*\chi)(\pm 1) = 0$ yields

$$\pi_r\chi - \pi_r^*\chi = \alpha_r\chi_r + \alpha_{r-1}\chi_{r-1}, \tag{A.1.5}$$

where

$$\chi_r = \frac{2}{(r+1)(r+2)}\,L'_{r+1} - \frac{2}{(r-1)r}\,L'_{r-1} \in \mathcal{P}^0_r(-1,1).$$

Multiplying (A.1.5) by $(1-\xi^2)(L'_{r-1}, L'_{r-2})$ and integrating gives

$$\begin{pmatrix}\alpha_r\\ \alpha_{r-1}\end{pmatrix} = -\frac{1}{4}\int_{-1}^{1}(\chi - \pi_r^*\chi)\begin{pmatrix}(2r-1)L'_{r-1}\\ (2r-3)L'_{r-2}\end{pmatrix}(1-\xi^2)\;d\xi.$$

Green's formula, together with Cauchy-Schwartz inequality and (A.1.2), shows that

$$\max(|\alpha_r|, |\alpha_{r-1}|) \leq \frac{C}{r^{-1/2+\nu}}\,\|\chi\|_{H^{1+\nu}(-1,1)}.$$

The triangular inequality applied to (A.1.5) yields

$$\begin{aligned}\|\chi - \pi_r\chi\|_{H^{1/2}_{00}(-1,1)} &\leq \|\chi - \pi_r^*\chi\|_{H^{1/2}_{00}(-1,1)}\\ &\quad + \frac{C}{r^{-1/2+\nu}}\,\|\chi\|_{H^{1+\nu}(-1,1)}\left(\|\chi_r\|_{H^{1/2}_{00}(-1,1)} + \|\chi_{r-1}\|_{H^{1/2}_{00}(-1,1)}\right).\end{aligned}$$

Using the bound (B.1) of Appendix B completes the proof. ∎

Let t be a bounded nonempty interval of length h_t, and denote π_r^t the L^2-orthogonal projection on $\mathcal{P}^0_r(t)$. Then we have the following corollary.

COROLLARY A.1.2. *For any $\nu \geq 0$, the following holds: $\forall\,\chi \in H^{1+\nu}(t) \cap H^1_0(t)$,*

$$\begin{aligned}\|\chi - \pi_r^t\chi\|_{L^2(t)} &\leq C\,\frac{h_t^{1+\alpha}}{r^{1+\nu}}\,\|\chi\|_{H^{1+\nu}(t)},\\ |\chi - \pi_r^t\chi|_{H^{1/2}_{00}(t)} &\leq C\,\frac{h_t^{1/2+\alpha}}{r^{1/2+\nu}}\,\sqrt{\log r}\,\|\chi\|_{H^{1+\nu}(t)},\end{aligned} \tag{A.1.6}$$

with $\alpha = \inf(\nu, r)$. The constant C depends neither on r nor on h_t.

The second operator we are interested in is denoted by $\tilde{\pi}_r$; it is the projection on $\mathcal{P}^0_r(-1,1)$ defined as follows, $\forall\,\chi \in L^2(-1,1)$:

$$\int_{-1}^{1}(\chi - \tilde{\pi}_r\chi)\,\psi_r\,d\xi = 0, \qquad \forall\,\psi_r \in \mathcal{P}^R_{r-1}(-1,1). \tag{A.1.7}$$

Similar to π_r, although it looks like an L^2-projection, the operator $\tilde{\pi}_r$ has a nice approximation estimate with respect to $H^{1/2}_{00}$-norm.

LEMMA A.1.3. *For any $\nu \geq 0$, the following holds:* $\forall \chi \in H^{1+\nu}(-1,1) \cap H_0^1(-1,1)$,

$$\|\chi - \tilde{\pi}_r \chi\|_{L^2(-1,1)} \leq \frac{C}{r^{1+\nu}} \|\chi\|_{H^{1+\nu}(-1,1)},$$

$$|\chi - \tilde{\pi}_r \chi|_{H_{00}^{1/2}(-1,1)} \leq C \frac{\sqrt{\log r}}{r^{1/2+\nu}} \|\chi\|_{H^{1+\nu}(-1,1)}.$$

PROOF. Proceeding as in the beginning of the previous proof, we have

$$\tilde{\pi}_r \chi - \pi_r^* \chi = \beta_r \chi_r + \beta_{r-1} \chi_{r-1}. \tag{A.1.8}$$

On one hand, the computation of the real β_{r-1} yields

$$\beta_{r-1} = -\frac{1}{4}(2r-3) \int_{-1}^{1} (\chi - \pi_r^* \chi) L'_{r-2} (1 - \xi^2) \, d\xi.$$

On the other hand, multiplying (A.1.8) by $(1+\xi)$ gives

$$\beta_r = -\frac{(r+2)(2r-1)}{(r-2)(2r+1)} \beta_{r-1}.$$

Then, we complete the proof as in Lemma A.1.1. ∎

REMARK A.1.1. The projection operator defined by taking $\mathcal{P}_{r-1}^L(-1,1)$ in (A.1.7) instead of $\mathcal{P}_{r-1}^R(-1,1)$ also satisfies the estimates of Lemma A.1.3.

For any t, a bounded nonempty interval of length h_t, $\tilde{\pi}_r^t$ is the projection operator on $\mathcal{P}_r^0(t)$ obtained from $\tilde{\pi}_r$ by a convenient translation and homotetic transformations. Then we have the following corollary.

COROLLARY A.1.4. *For any $\nu \geq 0$, it holds:* $\forall \chi \in H^{1+\nu}(t) \cap H_0^1(t)$,

$$\|\chi - \tilde{\pi}_r^t \chi\|_{L^2(t)} \leq C \frac{h_t^{1+\alpha}}{r^{1+\nu}} \|\chi\|_{H^{1+\nu}(t)},$$

$$|\chi - \tilde{\pi}_r^t \chi|_{H_{00}^{1/2}(t)} \leq C \frac{h_t^{1/2+\alpha}}{r^{1/2+\nu}} \sqrt{\log r}\, \|\chi\|_{H^{1+\nu}(t)},$$

with $\alpha = \inf(\nu, r)$. The constant C depends neither on r nor on h_t.

A.2. The *hp*-Mortar Finite-Element Projection

In order to simplify the presentation, we choose to work on the reference edge $A = (-1,1)$. The extension to an arbitrary edge $\Gamma_{k\ell}$ is made in an obvious way using a convenient translation and scaling. Henceforth, the mesh t^δ of A is associated with the subdivision $(\xi_i)_{0 \leq i \leq i^*}$ with $\xi_0 = -1$ and $\xi_{i*} = 1$ and $(t_i =]\xi_i, \xi_{i+1}[)_{0 \leq i \leq i^*-1}$ its elements. It is assumed to satisfy the (M) condition (3.1). The mortar projection, simply denoted π^δ ($\delta = (h,r)$ are the discretization parameters), is defined by (3.5) using spaces $W^\delta(A)$ and $M^\delta(A)$ constructed in the same way as $W^\delta(\Gamma_{k\ell})$ and $M^\delta(\Gamma_{k\ell})$. A first estimate on π^δ is given by Seshaiyer and Suri in [9], which is "suboptimal" (the bound is polluted by the factor $r^{3/4}$), $\forall \nu \geq 0$, $\forall \chi \in H^{1+\nu}(A)$,

$$\|\chi - \pi^\delta \chi\|_{H_{00}^{1/2}(A)} \leq C \frac{h^{1/2+\alpha}}{r^{1/2+\nu}} r^{3/4} \|\chi\|_{H^{1+\nu}(A)}, \tag{A.2.1}$$

with $\alpha = \min(\nu, r)$. The reason why they observed such a degradation of the accuracy of π^δ is that they first proved the following stability: $\forall \chi \in H_{00}^{1/2}(A)$,

$$\|\pi^\delta \chi\|_{H_{00}^{1/2}(A)} \leq C r^{3/4} \|\chi\|_{H_{00}^{1/2}(A)}, \tag{A.2.2}$$

and then they used it to derive (A.2.1). The continuity constant cannot be improved as observed in [9]; nevertheless, the estimate (A.2.1) is not optimal, and the approach that consists of evaluating (A.2.2) via (A.2.1) is not the correct one. The aim of this section is to reduce the extra-factor $r^{3/4}$ to $\sqrt{\log r}$ and then to prove the "quasioptimal" result, by producing a direct proof of the approximation error.

THEOREM A.2.1. *For any $\nu \geq 0$, the following estimate holds: $\forall \chi \in H^{1/2+\nu}(A)$,*

$$\|\chi - \pi^\delta \chi\|_{H_{00}^{1/2}(A)} \leq C \frac{h^\eta}{r^\nu} \sqrt{\log r}\, \|\chi\|_{H^{1/2+\nu}(A)} \tag{A.2.3}$$

with $\eta = \min(\nu, r)$.

For the proof, we adopt the methodology introduced in [29] for the h-finite elements and used in [9] for the hp-version. We need to split the operator π^δ into a sum of two operators. Their definitions require the following spaces (see [9]):

$$\begin{aligned}
W_1^\delta(A) &= \left\{\chi^\delta \in W^\delta(A),\ \chi^\delta(\xi_i) = 0,\ \forall i\,(0 \leq i \leq i^*)\right\},\\
M_1^\delta(A) &= \left\{\psi^\delta \in M^\delta(A),\ \psi^\delta(\xi_i) = 0,\ \forall i\,(1 \leq i \leq i^* - 1)\right\},\\
W_2^\delta(A) &= \left\{\chi^\delta \in W^\delta(A),\ \int_A \chi^\delta \psi^\delta\, d\xi = 0,\ \forall \psi^\delta \in M_1^\delta(A)\right\},\\
M_2^\delta(A) &= \left\{\psi^\delta \in M^\delta(A),\ \int_A \chi^\delta \psi^\delta\, d\xi = 0,\ \forall \chi^\delta \in W_1^\delta(A)\right\}.
\end{aligned}$$

We note that the functions $\psi^\delta \in M_1^\delta(A)$ do not necessarily vanish at $\{\xi_0, \xi_{i^*}\} = \{-1, 1\}$. The operators π_1^δ and π_2^δ are defined as in (3.5) using, respectively, the couples of spaces $(W_1^\delta(A), M_1^\delta(A))$ and $(W_2^\delta(A), M_2^\delta(A))$ instead of $(W^\delta(A), M^\delta(A))$. Due to the orthogonality of $(W_1^\delta(A), M_2^\delta(A))$ and $(W_2^\delta(A), M_1^\delta(A))$ with respect to the L^2-inner product, we have $\pi^\delta = \pi_1^\delta + \pi_2^\delta$. The next sections are dedicated to the analysis of π_1^δ and π_2^δ.

A.2.1. $H_{00}^{1/2}$-approximation result for π_1^δ

The operator π_1^δ has a good localization property; the restriction of $(\pi_1^\delta \varphi)$ to each element t_i $(0 \leq i \leq i^* - 1)$ depends only on $(\varphi_{|t_i})$. They are of two types: those acting within internal elements t_i $(1 \leq i \leq i^* - 2)$ are actually $\pi_r^{t_i}$, and those defined on the extremal elements are $\tilde{\pi}_r^{t_i}$, $i = 0$ or $(i^* - 1)$. Putting together the local error approximation of the Corollaries A.1.2 and A.1.4 provides a global estimate for π_1^δ.

LEMMA A.2.2. *For any $\nu \geq 0$, the following holds: $\forall \chi \in H^{1+\nu}(A)$ such that $\chi_{|t_i} \in H_0^1(t_i)$, $\forall i$ $(1 \leq i \leq i^* - 1)$,*

$$\|\chi - \pi_1^\delta \chi\|_{H_{00}^{1/2}(A)} \leq C \frac{h^{1/2+\alpha}}{r^{1/2+\nu}} \sqrt{\log r}\, \|\chi\|_{H^{1+\nu}(A)},$$

with $\alpha = \inf(\nu, r)$. The constant C depends neither on r nor on h.

PROOF. Letting $\chi \in H^{1+\nu}(A)$ such that $\chi(\xi_i) = 0$, $\forall i$ $(1 \leq i \leq i^* - 1)$, we have

$$\begin{aligned}
\|\chi - \pi_1^\delta \chi\|_{H_{00}^{1/2}(A)} &\leq C \left(\sum_{i=0}^{i^*-1} \|\chi - \pi_1^\delta \chi\|_{H_{00}^{1/2}(t_i)}^2\right)^{1/2}\\
&\leq C \left(\sum_{i=0}^{i^*-1} \left(\frac{h_t^{1/2+\alpha}}{r^{1/2+\nu}} \sqrt{\log r}\right)^2 \|\chi\|_{H^{1+\nu}(t_i)}^2\right)^{1/2}\\
&\leq C \frac{h^{1/2+\alpha}}{r^{1/2+\nu}} \sqrt{\log r} \left(\sum_{i=0}^{i^*-1} \|\chi\|_{H^{1+\nu}(t_i)}^2\right)^{1/2} \leq C \frac{h^{1/2+\alpha}}{r^{1/2+\nu}} \sqrt{\log r}\, \|\chi\|_{H^{1+\nu}(A)}. \quad \blacksquare
\end{aligned}$$

A.2.2. $H_{00}^{1/2}$-approximation results for π_2^δ

The space $W_2^\delta(A)$ is spanned by $(i^* - 1)$ nodal functions $(\chi_i)_{1 \leq i \leq (i^*-1)}$ (see [9,29]). Define the polynomial $\varphi_r \in \mathcal{P}_r(-1, 1)$ such that $\varphi_r(-1) = 0$, $\varphi_r(1) = 1$, and

$$\int_{-1}^{1} \varphi_r \psi_r\, d\xi = 0, \qquad \forall \psi_r \in \mathcal{P}_r^0(-1, 1).$$

Then $\varphi_r = \eta_r'$ where

$$\eta_r(\xi) = \frac{L_{r+1}(\xi)}{(r+1)(r+2)} + \frac{L_r(\xi)}{r(r+1)}$$

and $\tilde{\varphi}_r \in \mathcal{P}_r(-1,1)$ such that $\tilde{\varphi}_r(-1) = 0$, $\tilde{\varphi}_r(1) = 1$, and

$$\int_{-1}^{1} \tilde{\varphi}_r \psi_r \, d\xi = 0, \qquad \forall \psi_r \in \mathcal{P}_{r-1}^R(-1,1).$$

Then

$$\tilde{\varphi}_r = \frac{(1+\xi)L_r'(\xi)}{r(r+1)}.$$

The basis of $W_2^\delta(A)$ is set as follows: $(2 \le i \le i^* - 2)$,

$$\chi_i(\xi) = \begin{cases} \varphi_r\left(\dfrac{2\xi - \xi_i - \xi_{i-1}}{h_{i-1}}\right), & \text{on } t_{i-1}, \\ \varphi_r\left(\dfrac{\xi_i + \xi_{i+1} - 2\xi}{h_i}\right), & \text{on } t_i, \\ 0, & \text{otherwise,} \end{cases}$$

$$\chi_1(\xi) = \begin{cases} \tilde{\varphi}_r\left(\dfrac{2\xi - \xi_0 - \xi_1}{h_0}\right), & \text{on } t_0, \\ \varphi_r\left(\dfrac{\xi_1 + \xi_2 - 2\xi}{h_1}\right), & \text{on } t_1, \\ 0, & \text{otherwise,} \end{cases}$$

$$\chi_{i^*-1}(\xi) = \begin{cases} \tilde{\varphi}_r\left(\dfrac{\xi_{i^*-1} + \xi_{i^*} - 2\xi}{h_{i^*-1}}\right), & \text{on } t_{i^*-1}, \\ \varphi_r\left(\dfrac{2\xi - \xi_{i^*-1} - \xi_{i^*-2}}{h_{i^*-2}}\right), & \text{on } t_{i^*-2}, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, the space $M_2^\delta(A)$ coincides with $\mathrm{SP}\{\psi_1, \dots, \psi_{i^*-1}\}$. The basis involves $\psi_i = \chi_i$ $(2 \le i \le i^* - 2)$ and

$$\psi_1(\xi) = \begin{cases} \dfrac{h_0}{\xi - \xi_0}\chi_1(\xi), & \text{on } t_0, \\ \chi_1(\xi), & \text{otherwise,} \end{cases} \qquad \psi_{i^*-1}(\xi) = \begin{cases} \dfrac{h_{i^*-1}}{\xi_{i^*} - \xi}\chi_{i^*-1}(\xi), & \text{on } t_{i^*-1}, \\ \chi_{i^*-1}(\xi), & \text{otherwise.} \end{cases}$$

Let $\chi \in H_0^1(A)$ and $\pi_2^\delta \chi = \sum_{i=1}^{i^*-1} \alpha_i \chi_i$; introducing the vector $\underline{\chi}^\delta = (\alpha_1, \dots, \alpha_{i^*-1})$ and $\underline{\chi} = ((\chi, \psi_i)_{L^2(A)}, \dots, (\chi, \psi_{i^*-1})_{L^2(A)})$, we can write

$$(I + K)\underline{\chi}^\delta = D^{-1}\underline{\chi},$$

where $D = \mathrm{Diag}(D_1, \dots, D_{i^*-1})$ so that

$$D_i = (\chi_i, \psi_i)_{L^2(-1,1)} = \frac{h_i + h_{i-1}}{r(r+2)}, \qquad 2 \le i \le i^* - 2,$$

$$D_1 = (\chi_1, \psi_1)_{L^2(-1,1)} = \frac{h_0}{r(r+1)} + \frac{h_1}{r(r+2)},$$

$$D_{i^*-1} = (\chi_{i^*-1}, \psi_{i^*-1})_{L^2(-1,1)} = \frac{h_{i^*-1}}{r(r+1)} + \frac{h_{i^*-2}}{r(r+2)},$$

and K is a tridiagonal matrix with zero diagonal elements, the entries of which are computed by Seshaiyer and Suri in [9]. It is proven, therein, that

$$\|\pi_2^\delta \chi\|_{L^2(A)} \le C \left\| D^{-1/2} \underline{\chi} \right\|_E,$$

$$\left\| (\pi_2^\delta \chi)' \right\|_{L^2(A)} \le C \left\| D^{-3/2} \underline{\chi} \right\|_E,$$

where $\|.\|_E$ is the Euclidian norm of $\mathbb{R}^{(i^*-1)}$. The L^2-stability is valid for any kind of meshes while, so far, the H^1-stability is available only for (M)-meshes satisfying (3.1). From these inequalities, we deduce the following results.

LEMMA A.2.3. *Let $\chi \in H_0^1(A)$ be such that $\chi(\xi_i) = 0$, $\forall i$ $(0 \le i \le i^*)$. We have*

$$\|\pi_2^\delta \chi\|_{L^2(A)} \le C \frac{h}{r^{3/2}} |\chi|_{H^1(A)}, \tag{A.2.4}$$

$$|\pi_2^\delta \chi|_{H^1(A)} \le C r^{1/2} |\chi|_{H^1(A)}. \tag{A.2.5}$$

PROOF. The point is to estimate in a judicious way the terms $\|D^{-1/2}\underline{\chi}\|_E$ and $\|D^{-3/2}\underline{\chi}\|_E$. For any i $(1 \le i \le i^* - 1)$, we have

$$\underline{\chi}_i = (\chi, \psi_i)_{L^2(-1,1)} = \int_{t_{i-1}\cap t_i} \chi(\xi)\psi_i(\xi)\, d\xi.$$

First, we compute the integral on t_i $(1 \le i \le i^* - 2)$, using an integration by parts formula

$$\int_{t_i} \chi(\xi)\psi_i(\xi)\, d\xi = \int_{t_i} \chi(\xi)\eta_p'\left(\frac{\xi_i + \xi_{i+1} - 2\xi}{h_i}\right) d\xi = \frac{h_i}{2}\int_{t_i} \chi'(\xi)\eta_p\left(\frac{\xi_i + \xi_{i+1} - 2\xi}{h_i}\right) d\xi.$$

Cauchy-Schwartz inequality yields

$$\left|\int_{t_i} \chi(\xi)\psi_i(\xi)\, d\xi\right| \le \frac{h_i}{2} \|\chi'\|_{L^2(t_i)} \left\|\eta_p\left(\frac{\xi_i + \xi_{i+1} - 2\xi}{h_i}\right)\right\|_{L^2(t_i)} \le C \frac{h_i^{3/2}}{r^{5/2}} \|\chi'\|_{L^2(t_i)}.$$

Similarly, we have

$$\left|\int_{t_{i-1}} \chi(\xi)\psi_i(\xi)\, d\xi\right| \le C \frac{h_{i-1}^{3/2}}{r^{5/2}} \|\chi'\|_{L^2(t_{i-1})}.$$

There remains to look at the special cases of the extremal elements t_0 and t_{i^*-1}. First, observe that

$$\psi_1(\xi) = \frac{2}{r(r+1)} L_p'\left(\frac{2\xi - \xi_0 + \xi_1}{h_0}\right), \qquad \forall \xi \in t_0.$$

Following the same lines as above, we show that

$$\left|\int_{t_0} \chi(\xi)\psi_1(\xi)\, d\xi\right| \le C \frac{h_0^{3/2}}{r^{5/2}} \|\chi'\|_{L^2(t_0)}.$$

This gives

$$\left|\left(D^{-1/2}\underline{\chi}\right)_i\right| \le C \frac{r}{(h_i + h_{i-1})^{1/2}} \left(\frac{h_{i-1}^{3/2}}{r^{5/2}} \|\chi'\|_{L^2(t_{i-1})} + \frac{h_i^{3/2}}{r^{5/2}} \|\chi'\|_{L^2(t_i)}\right)$$
$$\le C \frac{h}{r^{3/2}} \|\chi'\|_{L^2(t_{i-1}\cup t_i)}.$$

Taking the sum of squares over i yields (A.2.4). On the other side, we have

$$\left|\left(D^{-3/2}\underline{\chi}\right)_i\right| \le C \frac{r^3}{(h_i + h_{i-1})^{3/2}} \left(\frac{h_{i-1}^{3/2}}{r^{5/2}} \|\chi'\|_{L^2(t_{i-1})} + \frac{h_i^{3/2}}{r^{5/2}} \|\chi'\|_{L^2(t_i)}\right)$$
$$\le C r^{1/2} \|\chi'\|_{L^2(t_{i-1}\cup t_i)}.$$

A sum of the squares over i provides (A.2.5) and completes the proof. ∎

LEMMA A.2.4. *Let $\chi \in H_0^1(A)$ be such that $\chi(\xi_i) = 0$, $\forall i$ $(0 \le i \le i^*)$. Then we have*

$$\|\pi_2^\delta \chi\|_{H_{00}^{1/2}(A)} \le C \frac{h^{1/2}}{r^{1/2}} |\chi|_{H^1(A)}. \tag{A.2.6}$$

PROOF. First of all, recall the compactness result of [20, Theorem 1.2.12]: for any real number $\varepsilon > 0$, the following inequality holds: $\forall \chi \in H_0^1(-1,1)$,

$$\|\chi\|_{H_{00}^{1/2}(-1,1)} \le \varepsilon |\chi|_{H^1(-1,1)} + \frac{C}{\varepsilon} \|\chi\|_{L^2(-1,1)}.$$

The constant $C > 0$ is independent of ε.

Let $\chi \in H_0^1(A)$ be given such that $\chi(\xi_i) = 0$, $\forall i$ $(0 \le i \le i^*)$. Choosing $\varepsilon = h^{1/2}/r$ in the previous inequality gives

$$\|\pi_2^\delta \chi\|_{H_{00}^{1/2}(A)} \le C \left(\frac{h^{1/2}}{r} |\pi_2^\delta \chi|_{H^1(A)} + \frac{r}{h^{1/2}} \|\pi_2^\delta \chi\|_{L^2(A)} \right).$$

By (A.2.4) and (A.2.5), we obtain (A.2.6). ∎

A.2.3. Proof of Theorem A.2.1

First, let us consider the projection operator studied in [12] and defined as follows: $\forall \chi \in H^1(A)$, $\breve{\pi}^\delta \chi \in W^\delta(A)$ is such that

$$\breve{\pi}^\delta \chi(\xi_i) = \chi(\xi_i), \qquad \forall i \ (0 \le i \le i^*),$$
$$\int_{t_i} (\chi - \breve{\pi}^\delta \chi)\, \psi^\delta \, d\xi = 0, \qquad \forall \psi^\delta \in \mathcal{P}_{r-2}(t).$$

We have the following estimate: for $\nu \ge 0$: $\forall \chi \in H^{1+\nu}(A)$,

$$\|\chi - \breve{\pi}^\delta \chi\|_{H^1(A)} \le C \frac{h^\alpha}{r^\nu} \|\chi\|_{H^{1+\nu}(A)},$$

where $\alpha = \min(r, \nu)$.

Then let $\chi \in H^{1+\nu}(A)$ be given. We write

$$\begin{aligned}\|\chi - \pi^\delta \chi\|_{H_{00}^{1/2}(A)} &= \left\|(\chi - \breve{\pi}^\delta \chi) - \pi^\delta (\chi - \breve{\pi}^\delta \chi)\right\|_{H_{00}^{1/2}(A)} \\ &\le \left\|(\chi - \breve{\pi}^\delta \chi) - \pi_1^\delta (\chi - \breve{\pi}^\delta \chi)\right\|_{H_{00}^{1/2}(A)} + \left\|\pi_2^\delta (\chi - \breve{\pi}^\delta \chi)\right\|_{H_{00}^{1/2}(A)}.\end{aligned}$$

Noticing that $(\chi - \breve{\pi}^\delta \chi)(\xi_i) = 0$, $\forall i$ $(0 \le i \le i^*)$, we can apply Lemmas A.2.2 and A.2.3, and we find

$$\begin{aligned}\|\chi - \pi^\delta \chi\|_{H_{00}^{1/2}(A)} &\le C \left(\frac{h^{1/2}}{r^{1/2}} \sqrt{\log(r)}\, \|\chi - \breve{\pi}^\delta \chi\|_{H^1(A)} + \frac{h^{1/2}}{r^{1/2}} \|\chi - \breve{\pi}^\delta \chi\|_{H^1(A)} \right) \\ &\le C \frac{h^{1/2+\alpha}}{r^{1/2+\nu}} \sqrt{\log(r)}\, \|\chi\|_{H^{1+\nu}(A)}.\end{aligned}$$

∎

APPENDIX B

The purpose of this appendix is to bound the $H_{00}^{1/2}(-1,1)$-norm of the function χ_r introduced in the proof of Lemma A.1.1.

LEMMA B.1. *We have*

$$\|\chi_r\|_{H^{1/2}_{00}(-1,1)} \leq C\,\frac{\sqrt{\log(r)}}{r}. \tag{B.1}$$

PROOF. Using the identity

$$L'_{r+1} = L'_{r-1} + (2r+1)L_r,$$

we deduce that

$$\chi_r = \frac{2(2r+1)}{(r+2)(r+1)}\left(L_r - \frac{2}{r(r-1)}\,L'_{r-1}\right),$$

and therefore,

$$\|\chi_r\|_{H^{1/2}(-1,1)} \leq Cr^{-1}\left(\|L_r\|_{H^{1/2}(-1,1)} + r^{-2}\left\|L'_{r-1}\right\|_{H^{1/2}(-1,1)}\right).$$

The inverse inequality (see [31]), combined with the fact that $\|L_r\|_{H^{1/2}(-1,1)} \leq C\sqrt{\log(r)}$ (see [39]), yields the partial result

$$\|\chi_r\|_{H^{1/2}(-1,1)} \leq C\,\frac{\sqrt{\log(r)}}{r}.$$

To bound the remainder term of $\|\chi_r\|_{H^{1/2}_{00}(-1,1)}$, we write

$$\int_{-1}^{1}\frac{\chi_p^2}{(1-\xi^2)}\,d\xi \leq Cr^{-2}\left\{\int_{-1}^{1}\frac{(1-L_r)^2}{1-\xi^2}\,d\xi + \int_{-1}^{1}\left(\frac{2}{p(p-1)}\,L'_{r-1}-1\right)^2\frac{d\xi}{1-\xi^2}\right\}.$$

The first term is bounded in by $C\log(r)$ in [39]. In order to work out the second, we resort to a Gauss-Lobatto numerical integration with r points and obtain

$$\int_{-1}^{1}\left(\frac{2}{(r-1)r}\,L'_{r-1}-1\right)^2\frac{d\xi}{1-\xi^2} = \sum_{j=1}^{r-2}\frac{1}{1-\left(\xi_j^{(r-1)}\right)^2}\,\rho_j^{(r-1)},$$

where $(\xi_j^{(r-1)})_{0\leq j\leq r-1}$ are the roots of $(1-\xi^2)L'_{r-1}$ and $(\rho_j^{(r-1)})_{0\leq j\leq r-1}$ the associated weights. Following the same line of argument as [39], this integral is bounded by $C\log(r)$. ∎

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