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A non-conforming finite element method for sub-meshing

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Abstract

In engineering applications, there is a constant need for solving problems using finite elements, over large and complicated domains. Such analysis is often accomplished by decomposing the global domain into several sub-domains and performing the analysis individually on each local sub-domain. The global analysis can then be realized by piecing together the information that is available from each sub-domain. *Non-conforming* finite elements are particularly useful in this regard. In this paper, we discuss such a technique for a model problem. We derive *necessary* and *sufficient* conditions in order for the mixed method formulation of such a non-conforming method to have a unique solution. Error estimates for the finite element solution are obtained. Finally, we present computational experiments to demonstrate that this non-conforming technique is a good candidate for finite element implementation.

Keywords: *h*-version; *p*-version; Finite elements; Non-conforming.

1. Introduction

Finite element methods have become an essential numerical and computing tool in most areas of mechanics. These techniques have spawned a wide variety of commercial *p* and *hp* version [2] programs whose success reveals the practical importance of these methods. However, even such techniques can become very

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challenging if the global domain is quite complex. A typical application will be the design of complex structural components (e.g. automobile components, the wing and fuselage of aircraft, etc.) being modeled simultaneously by different analysts. Non-conforming techniques help us to *piece* together the parts that are modeled independently, without sacrificing accuracy or efficiency. Moreover, one can also impose selective refinement only on those sub-components where it is required. Hence, in a variety of industrial applications, it is important to employ an efficient method that uses the (already) existing sub-meshes to solve the global system.

Fig. 1(a) shows an illustration of such a modeling complexity. Here we have a critical region ABCD where extra refinement is needed around the hole, whereas in the rest of the domain the refinement can be coarser. Using conforming techniques, one must perform transition modeling to coordinate the meshes around this critical region. The transition region usually includes triangular and/or distorted quadrilateral elements both of which contribute to modeling complexity and could, for severe distortions, result in solution inaccuracy. Therefore, methods which permit a high level of refinement in the local region (i.e., the critical region) and a coarser level of refinement in the global region (i.e., the region away from the critical region) without requiring transition modeling are highly desirable. Such an ideal situation is shown in the non-conforming mesh of Fig. 1(b). In this case the meshes from the two different levels of refinement do not match, which then allows the sub-meshes to be constructed independently. Hence the finite elements in all regions may be constructed as regularly shaped as possible. This coupling approach may also

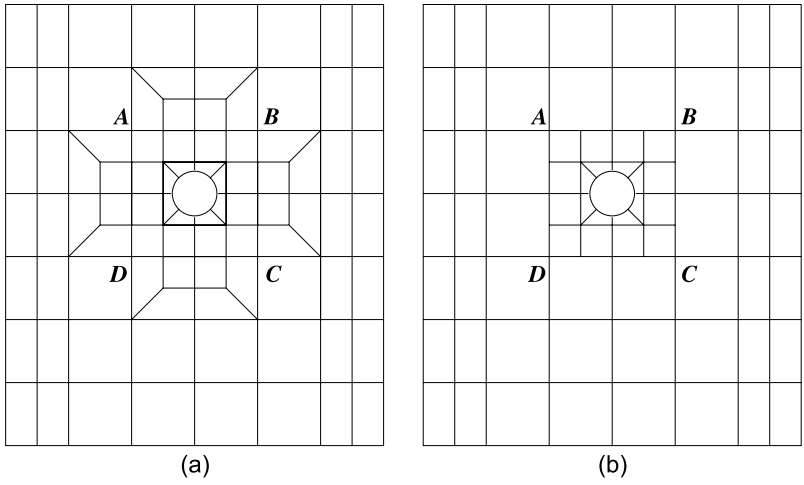

Fig. 1. (a) Conforming and (b) non-conforming mesh.

be used to connect large sub-structures such as wing and fuselage structures that may have been modeled by different analysts in different groups or organizations. Transition modeling can become quite complex, tedious and expensive in such cases because the finite element nodes of each component at the common interface are not, in general, coincident. The use of non-conforming methods (at the sub-domain level) can help in this regard. Moreover, it is natural to distribute independently modeled sub-domains across a parallel computer architecture via this sub-meshing technique. Some related parallel domain decomposition approaches can be found in [4,9].

Briefly, in these methods the inter-domain continuity requirement,

$$s^i - s^j = 0 \text{ on } \Gamma_{ij}, \tag{1.1}$$

is enforced by using one or more interface spaces in a weak sense. Here $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$ is the interface between two sub-domains Ω_i and Ω_j and s^i, s^j are the values of the test or trial function on Γ_{ij} from the two sides. An example of such a technique is the mortar finite element method [3,5,6,16]. Some other examples can be found in [8,11,18] (some of these are defined only at the inter-element, rather than the inter-sub-domain level).

Another approach would be to define an interface space corresponding to the trace of the true solution on Γ_{ij}, which forms the field t. We then introduce two auxiliary spaces to deal with the constraints (see Fig. 2),

$$s^i - t = 0, \quad s^j - t = 0 \text{ on } \Gamma_{ij}. \tag{1.2}$$

Here, n_i and n_j are the respective outward normals to Ω_i and Ω_j. Variants of this idea have been proposed in the literature in [1,7]. We point out that a version of the method in [1] has been implemented in the commercial *hp* program MSC-NASTRAN (see [12,13] for more *hp* computational results). In this paper, we present the mixed method formulation and implementation of

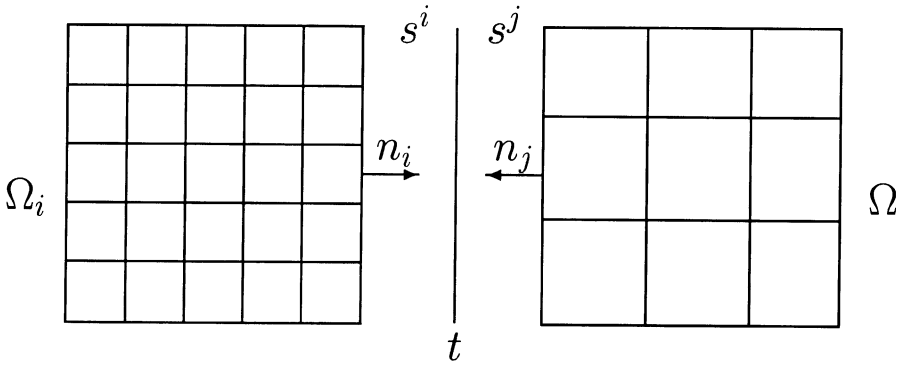

Fig. 2. Non-conforming domain decomposition of Ω.

such a non-conforming technique for a model problem. This method allows more flexibility in the choice of the approximation spaces and therefore, has obvious advantages over other coupling methods. We will also show, that the mortar finite element method can be derived as a special case of this method.

2. The model problem and its discretization

Consider the following second-order model elliptic problem,

$$-\Delta Z = f \quad \text{on } \Omega, \tag{2.1}$$

with the boundary conditions,

$$Z = 0 \quad \text{on } \partial\Omega_{\mathrm{D}}, \qquad \frac{\partial Z}{\partial n} = g \quad \text{on } \partial\Omega_{\mathrm{N}}, \tag{2.2}$$

where Ω is a polygonal domain with boundary $\partial\Omega = \overline{\partial\Omega}_{\mathrm{D}} \cup \overline{\partial\Omega}_{\mathrm{N}}$ ($\partial\Omega_{\mathrm{D}} \cap \partial\Omega_{\mathrm{N}} = \emptyset, \partial\Omega_{\mathrm{D}} \neq \emptyset$). Although we consider (2.1), all our results hold for more general problems as well, e.g. linear elasticity.

Let $H^k(\Omega)$ denote the usual Sobolev space of functions with k generalized derivatives on Ω, with $L_2(\Omega) = H^0(\Omega)$. Both $\|\cdot\|_{k,A}$ and $\|\cdot\|_{\mathrm{H}^k(\mathrm{A})}$ will be used to denote the norm of $H^k(\Omega)$. We then define the energy space,

$$H^1_{\mathrm{D}}(\Omega) \stackrel{\mathrm{def}}{=} \{\mathrm{Z} \in \mathrm{H}^1(\Omega) | \mathrm{Z} = 0 \quad \text{on } \partial\Omega_{\mathrm{D}}\}.$$

Note that the definition of these spaces can be extended to non-integer values of k by interpolation. For instance, for any interval $\mathscr{I}$, $H^{1/2}_{00}(\mathscr{I})$ can be the space obtained between $L_2(\mathscr{I})$ and $H^1_{\mathrm{D}}(\mathscr{I})$. With these definitions, the form (2.1) is equivalent to the following variational form: Find $Z \in H^1_{\mathrm{D}}(\Omega)$ satisfying, for all $v \in H^1_{\mathrm{D}}(\Omega)$,

$$a(Z, v) \stackrel{\mathrm{def}}{=} \int_{\Omega} \nabla \mathrm{Z} \cdot \nabla \mathrm{v}\, \mathrm{dx} = \int_{\Omega} \mathrm{fv}\, \mathrm{dx} + \int_{\partial\Omega_{\mathrm{N}}} \mathrm{gv}\, \mathrm{ds} \stackrel{\mathrm{def}}{=} \mathrm{F(v)}. \tag{2.3}$$

In order to define the finite element discretization, we assume Ω is the union of S non-overlapping polygonal sub-domains $\{\Omega_i\}_{i=1}^{S}$ such that $\partial\Omega_i \cap \partial\Omega_j (i < j)$ is either empty, a vertex, or an entire edge of Ω_i and Ω_j. For simplicity, let us assume that we are solving this problem over a domain Ω, which is divided into two sub-domains Ω_1 and Ω_2 with their common interface denoted by Γ_{12} (see Fig. 3).

Then our problem of finding the continuous solution Z which satisfies (2.1) and (2.2) becomes solving for s^i (the interior solution variable in each domain Ω_i), which satisfies for $i = 1, 2$,

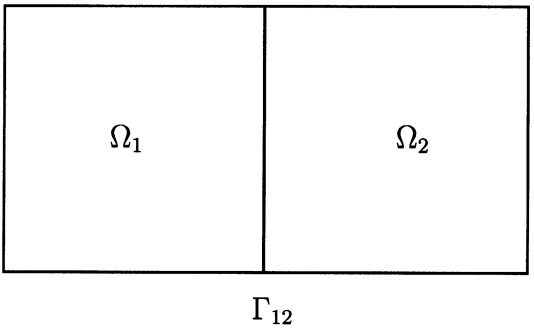

Fig. 3. Domain decomposition of Ω.

$$\begin{aligned} &-\Delta s^i = f_i \quad \text{on } \Omega_i, \\ &s^i = 0 \quad \text{on } \partial\Omega_{\mathrm{D}} \cap \partial\Omega_i, \\ &\frac{\partial s^i}{\partial n_i} = g \quad \text{on } \partial\Omega_{\mathrm{N}} \cap \partial\Omega_i \end{aligned} \tag{2.4}$$

along with the continuity condition enforced on the trace of the solution on Γ_{12} given by, $s^i = t$ on Γ_{12}. Here, t is a new unknown we introduce, called the interface displacement on Γ_{12}. Note that, we can rewrite this condition in integral form as,

$$\int_{\Gamma_{12}} s^i \psi \, \mathrm{d}s = \int_{\Gamma_{12}} t\psi \, \mathrm{d}s, \tag{2.5}$$

where ψ is any function in $H^{-1/2}(\Gamma_{12})$. We now introduce the test functions v_i (corresponding to each domain Ω_i) such that, $v_i = 0$ on $\partial\Omega_{\mathrm{D}} \cap \partial\Omega_i$ and write (2.4) in variational form. This then gives, for $i = 1, 2$,

$$\int_{\Omega_i} \nabla s^i \cdot \nabla v_i \, \mathrm{d}x - \int_{\Gamma_{12}} \frac{\partial s^i}{\partial n_i} v_i \, \mathrm{d}s = \int_{\Omega_i} f_i v_i \, \mathrm{d}x + \int_{\partial\Omega_i \cap \partial\Omega_{\mathrm{N}}} g_i v_i \, \mathrm{d}s. \tag{2.6}$$

Note that, since the solution is smooth in the interior of Ω,

$$\frac{\partial s^1}{\partial n_1} + \frac{\partial s^2}{\partial n_2} = 0.$$

Introducing a new unknown variable $n^i = -(\partial s^i / \partial n_i)$ $(i = 1, 2)$, the above equation can also be written in integral form as,

$$\int_{\Gamma_{12}} (n^1 + n^2)\mu \, \mathrm{d}s = 0 \tag{2.7}$$

for all $\mu \in H^{1/2}(\Gamma_{12})$.

To discretize (2.5)–(2.7) by the finite element method, we choose finite dimensional spaces for the unknown $u = (s^1, s^2, n^1, n^2, t)$. For $i = 1, 2$, let s^i,

$v_i \in S^i \subset H^1_{\mathrm{D}}(\Omega_i)$, n^i, $\psi_i \in N^i \subset H^{-1/2}(\Gamma_{12})$ and t, $\mu \in T \subset H^{1/2}(\Gamma_{12})$. Let $s^i|_{\Gamma_{12}}$ be the trace space for s^i. Then our problem can be stated as follows: Find $u = (s^1, s^2, n^1, n^2, t) \in S^1 \times S^2 \times N^1 \times N^2 \times T$ such that for all $(v_1, v_2, \psi_1, \psi_2, \mu) \in S^1 \times S^2 \times N^1 \times N^2 \times T$

$$\int_{\Omega_1} \nabla s^1 \cdot \nabla v_1 \, \mathrm{d}x + \int_{\Gamma_{12}} n^1 v_1 \, \mathrm{d}s = F_1(v_1), \tag{2.8}$$

$$\int_{\Omega_2} \nabla s^2 \cdot \nabla v_2 \, \mathrm{d}x + \int_{\Gamma_{12}} n^2 v_2 \, \mathrm{d}s = F_2(v_2), \tag{2.9}$$

$$\int_{\Gamma_{12}} s^1 \psi_1 \, \mathrm{d}s - \int_{\Gamma_{12}} t \psi_1 \, \mathrm{d}s = 0, \tag{2.10}$$

$$\int_{\Gamma_{12}} s^2 \psi_2 \, \mathrm{d}s - \int_{\Gamma_{12}} t \psi_2 \, \mathrm{d}s = 0, \tag{2.11}$$

$$-\int_{\Gamma_{12}} (n^1 + n^2)\mu \, \mathrm{d}s = 0, \tag{2.12}$$

where

$$F_i(v_i) = \int_{\Omega_i} f_i v_i \, \mathrm{d}x + \int_{\partial\Omega_i \cap \partial\Omega_{\mathrm{N}}} g v_i \, \mathrm{d}s.$$

Remark 2.1. If we choose the spaces $N^1 = N^2 = N$, then in (2.10) and (2.11) we can choose $\psi_1 = \psi_2 = \chi \in N$. Subtracting (2.11) from (2.10) then gives,

$$\int_{\Gamma_{12}} (s^1 - s^2)\chi = 0 \ \forall \chi \in N.$$

Suppose moreover that N is a space of piecewise polynomials on a given triangulation of the edge Γ_{12}. Also, set $n = n^1 = -n^2$. Then (2.12) is satisfied, and (2.8)–(2.12) becomes,

$$\int_{\Omega_1} \nabla s^1 \cdot \nabla v_1 \, \mathrm{d}x + \int_{\Gamma_{12}} n v_1 \, \mathrm{d}s = F_1(v_1),$$
$$\int_{\Omega_2} \nabla s^2_{\mathrm{N}} \cdot \nabla v_2 \, \mathrm{d}x - \int_{\Gamma_{12}} n v_2 \, \mathrm{d}s = F_1(v_2),$$
$$\int_{\Gamma_{12}} (s^1 - s^2)\chi \, \mathrm{d}s = 0.$$

Hence adding the above equations, our problem is simplified to finding (s^1, s^2, n). It can be shown that for a simple choice of approximation of the normal spaces, the above mixed formulation yields the mortar finite element method [15,17].

Let us now define the finite dimensional spaces, to be the following,

$$S^i = \mathrm{Span}\{s_1^i, s_2^i, \ldots, s_{p_i}^i\} \quad \text{for } i = 1,2,$$
$$N^i = \mathrm{Span}\{n_1^i, n_2^i, \ldots, n_{q_i}^i\} \quad \text{for } i = 1,2,$$
$$T = \mathrm{Span}\{t_1, t_2, \ldots, t_r\},$$

where s_j^i, n_j^i, t_j are piecewise polynomials. (Note that, s_j^i vanish on $\partial\Omega_D \cap \partial\Omega_i$.) Then we can express, for $i = 1,2$,

$$s^i = \sum_{j=1}^{p_i} \alpha_j^i s_j^i, \quad n^i = \sum_{j=1}^{q_i} \beta_j^i n_j^i, \quad t = \sum_{j=1}^{r} \delta_j t_j. \tag{2.13}$$

Since, $(v_1, v_2, \psi_1, \psi_2, \mu) \in S^1 \times S^2 \times N^1 \times N^2 \times T$, we can choose for $i = 1,2$,

$$v_i = s_k^i, \quad \psi_i = n_k^i, \quad \mu = t_k. \tag{2.14}$$

Substituting (2.13) and (2.14) in the system (2.8)–(2.12) gives,

$$\sum_{j=1}^{p_1} \alpha_j^1 \int_{\Omega_1} \nabla s_j^1 \cdot \nabla s_k^1 \,\mathrm{d}x + \sum_{j=1}^{q_1} \beta_j^1 \int_{\Gamma_{12}} n_j^1 s_k^1 \,\mathrm{d}s = F_1(s_k^1), \tag{2.15}$$

$$\sum_{j=1}^{p_2} \alpha_j^2 \int_{\Omega_2} \nabla s_j^2 \cdot \nabla s_k^2 \,\mathrm{d}x + \sum_{j=1}^{q_2} \beta_j^2 \int_{\Gamma_{12}} n_j^2 s_k^2 \,\mathrm{d}s = F_2(s_k^2), \tag{2.16}$$

$$\sum_{j=1}^{p_1} \alpha_j^1 \int_{\Gamma_{12}} s_j^1 n_k^1 \,\mathrm{d}s - \sum_{j=1}^{r} \delta_j \int_{\Gamma_{12}} t_j n_k^1 \,\mathrm{d}s = 0, \tag{2.17}$$

$$\sum_{j=1}^{p_2} \alpha_j^2 \int_{\Gamma_{12}} s_j^2 n_k^2 \,\mathrm{d}s - \sum_{j=1}^{r} \delta_j \int_{\Gamma_{12}} t_j n_k^2 \,\mathrm{d}s = 0, \tag{2.18}$$

$$-\sum_{j=1}^{q_1} \beta_j^1 \int_{\Gamma_{12}} n_j^1 t_k \,\mathrm{d}s - \sum_{j=1}^{q_2} \beta_j^2 \int_{\Gamma_{12}} n_j^2 t_k \,\mathrm{d}s = 0. \tag{2.19}$$

Let A_i, B_i, C_i, $i = 1,2$ be the matrices defined by,

$$A_i^{k,j} = \int_{\Omega_i} \nabla s_j^i \cdot \nabla s_k^i \,\mathrm{d}x, \quad B_i^{k,j} = \int_{\Gamma_{12}} n_j^i s_k^i \,\mathrm{d}s, \quad C_i^{k,j} = \int_{\Gamma_{12}} t_j \, n_k^i \,\mathrm{d}s \tag{2.20}$$

and let $\vec{F}^i$, $i = 1,2$ be vectors defined as $F_k^i = F_i(s_k^i)$. Let us also define the vectors,

$$\overrightarrow{\alpha^i} = [\alpha_1^i \alpha_2^i \ldots \alpha_{p_i}^i], \quad \overrightarrow{\beta^i} = [\beta_1^i \beta_2^i \ldots \beta_{q_i}^i], \quad \vec{\delta} = [\delta_1 \delta_2 \ldots \delta_r].$$

Then the system of Eqs. (2.15)–(2.19) reduces to,

$$P\vec{x} = \vec{f}, \tag{2.21}$$

where the solution $\vec{x} = [\,\overrightarrow{\alpha^1} \quad \overrightarrow{\alpha^2} \quad \overrightarrow{\beta^1} \quad \overrightarrow{\beta^2} \quad \vec{\delta}\,]^{\mathrm{T}}$, the right hand side vector $\vec{f} = [\,\vec{F}^1 \quad \vec{F}^2 \quad 0 \quad 0 \quad 0\,]$ and the general stiffness matrix P is given by,

$$P = \begin{bmatrix} A_1 & 0 & B_1 & 0 & 0 \\ 0 & A_2 & 0 & B_2 & 0 \\ (B_1)^{\mathrm{T}} & 0 & 0 & 0 & C_1 \\ 0 & (B_2)^{\mathrm{T}} & 0 & 0 & C_2 \\ 0 & 0 & (C_1)^{\mathrm{T}} & (C_2)^{\mathrm{T}} & 0 \end{bmatrix}. \tag{2.22}$$

Let us now derive the *sufficient* conditions for (2.21) to have a unique solution. Note that since A_i $(i = 1, 2)$ are square, invertible and positive definite matrices as they correspond to a regular finite element problem. Therefore,

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

is a positive definite matrix. Taking the symmetric *Schur complement* [10,14], with respect to A_1 and A_2 yields

$$\widetilde{P} = \begin{bmatrix} -B_1^{\mathrm{T}} A_1^{-1} B_1 & 0 & C_1 \\ 0 & -B_2^{\mathrm{T}} A_2^{-1} B_2 & C_2 \\ (C_1)^{\mathrm{T}} & (C_2)^{\mathrm{T}} & 0 \end{bmatrix}. \tag{2.23}$$

Note that the matrix P is invertible if and only if $\widetilde{P}$ is invertible.

Lemma 2.1. *Consider the matrix Q defined by*

$$Q = \begin{bmatrix} B & C \\ C^{\mathrm{T}} & 0 \end{bmatrix},$$

where B is a semidefinite square matrix. The matrix Q is invertible if and only if,

(A) C has full column rank and,
(B) $C^{\mathrm{T}}x = 0$ and $Bx = 0$ implies $x = 0$.

For details of the proof of Lemma 2.1, please refer to [10,14].

In our case note that the matrix B is a negative semidefinite block matrix,

$$B = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}, \tag{2.24}$$

where $G_i = -(B_i)^{\mathrm{T}}(A_i)^{-1}B_i$.

Therefore using Lemma 2.1 and the structure of matrix B in (2.24), we have the following theorem.

Theorem 2.1. *The matrix P in* (2.22) *is invertible if and only if,*

(A) $\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ has full column rank and,

(B) $\sum_{i=1}^{2} C_i^{\mathrm{T}} x^i = 0$ and $[B_1 x^1, B_2 x^2] = [0, 0]$ implies $[x^1, x^2] = [0, 0]$.

Let us interpret these conditions geometrically. Condition (A) in Theorem 2.1 implies that the normal spaces N^1, N^2 and the trace space T satisfy,

$$(N^1 + N^2)^{\perp} \cap T = \emptyset.$$

This relation between the spaces N^i and T required for $\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ to have full column rank, can be re-expressed as follows:

Condition (A1). For each non-zero $t \in T$, there exists $n^1 \in N^1$ *or* $n^2 \in N^2$ such that, $\int_{\Gamma_{12}} tn^1 \, ds > 0$ or $\int_{\Gamma_{12}} tn^2 \, ds > 0$.

Condition (B) in Theorem 2.1 means that if $n^i \perp (S^i|_{\Gamma_{12}})$ with $(n^1 + n^2) \perp T$ then we have $n^1 = 0$ and $n^2 = 0$. This translates to the following condition:

Condition (B1). For any non-zero pair $(n^1, n^2) \in N^1 \otimes N^2$ there either exists $s^i \in S^i|_{\Gamma_{12}}$, such that,

$$\int_{\Gamma_{12}} n^i s^i \, ds > 0 \quad \text{for } i = 1 \text{ or } i = 2 \tag{2.25}$$

or there exists $t \in T$ such that

$$\int_{\Gamma_{12}} (n^1 + n^2) t \, ds > 0. \tag{2.26}$$

Remark 2.2. Note that in [1], the following conditions have been presented for invertibility:

$$\dim(\mathrm{T}) \leqslant \max(\dim(N^1), \dim(N^2)), \tag{2.27}$$

$$\dim(N^j) \leqslant \dim(S^j|_{\Gamma_{12}}) \quad (j = 1, 2). \tag{2.28}$$

Although these might be *necessary* for certain classes of subspaces, they are clearly not equivalent to conditions (A1), (B1) and are *not* sufficient or necessary in general.

3. Error estimates

We will now prove that the error in the solution for this mixed method is bounded by an *approximation* error term $e_A(u)$ and a *consistency* error term $e_C(u)$. Let us now define the spaces,

$$\mathscr{X} = H^1_{\mathrm{D}}(\Omega_1) \times H^1_{\mathrm{D}}(\Omega_2) \times H^{-1/2}(\Gamma_{12}) \times H^{-1/2}(\Gamma_{12}) \times H^{1/2}(\Gamma_{12}),$$
$$X = S^1 \times S^2 \times N^1 \times N^2 \times T.$$

Let $\tilde{u} \in \mathscr{X}$ correspond to the classical solution of (2.1). Note that this exact solution $\tilde{u} = (\tilde{s}^1, \tilde{s}^2, \tilde{n}^1, \tilde{n}^2, \tilde{t})$ satisfies the following equations, for all $\tilde{v} = (\tilde{v}_1, \tilde{v}_2, \tilde{\psi}_1, \tilde{\psi}_2, \tilde{\mu}) \in \mathscr{X}$.

$$\int_{\Omega_1} \nabla \tilde{s}^1 \cdot \nabla \tilde{v}_1 \, \mathrm{d}x + \int_{\Gamma_{12}} \tilde{n}^1 \tilde{v}_1 \, \mathrm{d}s = F_1(\tilde{v}_1) \quad \forall \tilde{v}_1 \in H^1_{\mathrm{D}}(\Omega_1), \tag{3.1}$$

$$\int_{\Omega_2} \nabla \tilde{s}^2 \cdot \nabla \tilde{v}_2 \, \mathrm{d}x + \int_{\Gamma_{12}} \tilde{n}^2 \tilde{v}_2 \, \mathrm{d}s = F_2(\tilde{v}_2) \quad \forall \tilde{v}_2 \in H^1_{\mathrm{D}}(\Omega_2), \tag{3.2}$$

$$\int_{\Gamma_{12}} \tilde{s}^1 \, \tilde{\psi}_1 \, \mathrm{d}s - \int_{\Gamma_{12}} \tilde{t} \tilde{\psi}_1 \, \mathrm{d}s = 0 \quad \forall \tilde{\psi}_1 \in H^{-1/2}(\Gamma_{12}), \tag{3.3}$$

$$\int_{\Gamma_{12}} \tilde{s}^2 \tilde{\psi}_2 \, \mathrm{d}s - \int_{\Gamma_{12}} \tilde{t} \tilde{\psi}_2 \, \mathrm{d}s = 0 \quad \forall \tilde{\psi}_2 \in H^{-1/2}(\Gamma_{12}), \tag{3.4}$$

$$-\int_{\Gamma_{12}} (\tilde{n}^1 + \tilde{n}^2) \tilde{\mu} \, \mathrm{d}s = 0 \quad \forall \tilde{\mu} \in H^{1/2}(\Gamma_{12}). \tag{3.5}$$

We recall that the approximate solution $u = (s^1, s^2, n^1, n^2, t) \in X$ satisfies Eqs. (2.8)–(2.12). Let us define,

$$\left(\sum_{i=1}^{2} \|\nabla v^i\|^2_{0,\Omega_i} \right)^{1/2} = |v|_{1,S}.$$

We then have the following error estimate.

Theorem 3.1. *Let X_{N} be the subspace of X satisfying the homogeneous equations* (2.10) *and* (2.11). *Let $s = (s_1, s_2) \in X_{\mathrm{N}}$ and $\mu \in T$. Then for any $\tilde{s} = (\tilde{s}_1, \tilde{s}_1)$ we have*,

$$|\tilde{s} - s|_{1,S} \leqslant C(e_A(\tilde{s}) + e_C(\tilde{s})),$$

where $e_A(\tilde{s})$ and $e_C(\tilde{s})$ denote the approximation error and consistency error respectively which are given by,

$$e_A(\tilde{s}) = \inf_{w=(w_1,w_2)\in X_{\mathrm{N}}} |\tilde{s} - w|_{1,S},$$

$$e_C(\tilde{s}) = \sup_{v=(v_1,v_2)\in X_{\mathrm{N}}} \inf_{(\xi_1,\xi_2)\in N^1\times N^2} \frac{\sum_{i=1}^{2} \int_{\Gamma_{12}} (n^i - \xi_i)(v_i - \mu) \, \mathrm{d}s}{|v|_{1,S}}.$$

Proof. Using Eqs. (2.8), (2.9), (3.1) and (3.2) for any $(w^1, w^2) \in X_{\mathrm{N}}$ and $(v_1, v_2) \in X_{\mathrm{N}}$ we have,

$$\sum_{i=1}^{2}\left(\int_{\Omega_i}\nabla(s^i-w^i)\cdot\nabla v_i\,\mathrm{d}x+\int_{\Gamma_{12}}n^i v_i\,\mathrm{d}s\right)$$
$$=\sum_{i=1}^{2}\left(\int_{\Omega_i}\nabla(\tilde{s}^i-w^i)\cdot\nabla v_i\,\mathrm{d}x+\int_{\Gamma_{12}}\tilde{n}^i v_i\,\mathrm{d}s\right). \tag{3.6}$$

Note that,

$$\int_{\Gamma_{12}}v_1 n^1\,\mathrm{d}s+\int_{\Gamma_{12}}v_2 n^2\,\mathrm{d}s=\int_{\Gamma_{12}}t(n^1+n^2)\,\mathrm{d}s=0, \tag{3.7}$$

where we have used (2.12). Now let $\mu\in T\subset H^{1/2}(\Gamma_{12})$. Then (3.5) gives,

$$\int_{\Gamma_{12}}\tilde{n}^1 v_1\,\mathrm{d}s+\int_{\Gamma_{12}}\tilde{n}^2 v_2\,\mathrm{d}s=\int_{\Gamma_{12}}\tilde{n}^1(v_1-\mu)\,\mathrm{d}s+\int_{\Gamma_{12}}\tilde{n}^2(v_2-\mu)\,\mathrm{d}s. \tag{3.8}$$

Now, for all $\xi=(\xi^1,\xi^2)\in N^1\times N^2$, (2.10) and (2.11) give,

$$\int_{\Gamma_{12}}(v_i-\mu)\xi^i\,\mathrm{d}s=0\quad\text{for } i=1,2.$$

Using the above Eq. (3.8) can then be written as,

$$\int_{\Gamma_{12}}\tilde{n}^1 v_1\,\mathrm{d}s+\int_{\Gamma_{12}}\tilde{n}^2 v_2\,\mathrm{d}s=\sum_{i=1}^{2}\int_{\Gamma_{12}}(\tilde{n}^i-\xi^i)(v_i-\mu)\,\mathrm{d}s. \tag{3.9}$$

Substituting (3.7) and (3.9) in (3.6) we have,

$$\sum_{i=1}^{2}\int_{\Omega_i}\nabla(s^i-w^i)\cdot\nabla v_i\,\mathrm{d}x=\sum_{i=1}^{2}\int_{\Omega_i}\nabla(\tilde{s}^i-w^i)\cdot\nabla v_i\,\mathrm{d}x$$
$$+\sum_{i=1}^{2}\int_{\Gamma_{12}}(\tilde{n}^i-\xi^i)(v_i-\mu)\,\mathrm{d}s. \tag{3.10}$$

Dividing by $\left(\sum_{i=1}^{2}\|\nabla v_i\|^2_{0,\Omega_i}\right)^{1/2}$ and taking the supremum over all $v=(v_1, v_2)\in X_{\mathrm{N}}$, Eq. (3.10) becomes,

$$\sup_{v\in X_{\mathrm{N}}}\frac{\sum_{i=1}^{2}\int_{\Omega_i}\nabla(s^i-w^i)\cdot\nabla v_i\,\mathrm{d}x}{\left(\sum_{i=1}^{2}\|\nabla v_i\|^2_{0,\Omega_i}\right)^{1/2}}=\sup_{v\in X_{\mathrm{N}}}\frac{\sum_{i=1}^{2}\int_{\Omega_i}\nabla(\tilde{s}^i-w^i)\cdot\nabla v_i\,\mathrm{d}x}{\left(\sum_{i=1}^{2}\|\nabla v_i\|^2_{0,\Omega_i}\right)^{1/2}}+e(\tilde{s}), \tag{3.11}$$

where $e(\tilde{s})$ is given by,

$$e(\tilde{s}) = \sup_{v \in X_N} \frac{\sum_{i=1}^{2} \int_{\Gamma_{12}} (\tilde{n}^i - \xi^i)(v_i - \mu)\,\mathrm{d}s}{\left(\sum_{i=1}^{2} \|\nabla v_i\|^2_{0,\Omega_i} \right)^{1/2}}.$$

Since $s = (s^1, s^2) \in X_N$, we may choose $v^i = s^i - w^i$ for $i = 1, 2$, so that, the left hand side of (3.11) becomes,

$$\sup_{v \in X_N} \frac{\sum_{i=1}^{2} \int_{\Omega_i} \nabla(s^i - w^i) \cdot \nabla v_i \,\mathrm{d}x}{\left(\sum_{i=1}^{2} \|\nabla v_i\|^2_{0,\Omega_i} \right)^{1/2}} \geqslant \frac{\sum_{i=1}^{2} \int_{\Omega_i} \nabla(s^i - w^i) \cdot \nabla(s^i - w^i) \,\mathrm{d}x}{\left(\sum_{i=1}^{2} \|\nabla(s^i - w^i)\|^2_{0,\Omega_i} \right)^{1/2}}$$

$$= |s - w|_{1,S}. \tag{3.12}$$

Moreover,

$$\sup_{v \in X_N} \frac{\sum_{i=1}^{2} \int_{\Omega_i} \nabla(\tilde{s}^i - w^i) \cdot \nabla v_i \,\mathrm{d}x}{\left(\sum_{i=1}^{2} \|\nabla v_i\|^2_{0,\Omega_i} \right)^{1/2}} \leqslant |\tilde{s} - w|_{1,S}. \tag{3.13}$$

Now using Eqs. (3.11)–(3.13), and the triangle inequality,

$$|\tilde{s} - s|_{1,S} \leqslant |\tilde{s} - w|_{1,S} + |s - w|_{1,S} \leqslant 2|\tilde{s} - w|_{1,S} + e(\tilde{s})$$
$$\leqslant C(e_A(\tilde{s}) + e_C(\tilde{s})). \qquad \square$$

4. Numerical results

In this section, we demonstrate the performance of the numerical technique discussed in Section 2. Our calculations are performed for our model problem (2.1) and (2.2) on the domain in Fig. 4. We partition this domain into two rectangular domains Ω_1 and Ω_2, by the interface AO. We prescribe Neumann

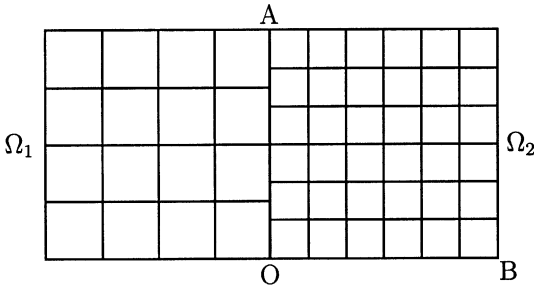

Fig. 4. Decomposition of Ω into Ω_1 ($m = 4$) and Ω_2 ($n = 6$).

boundary conditions, where $\partial\Omega_N = \partial\Omega$, with uniqueness maintained by imposing the condition $u = 0$ at the single point B.

We choose the right hand side f in (2.1) such that our exact solution is,

$$Z(x,y) = (x^2 + y^2)^2 - 1.$$

The non-conforming method is implemented as a *mixed* method as described earlier. For programming convenience and illustration, we restrict our meshes to *tensor product* meshes with Ω_1 divided into m^2 rectangles (by taking uniformly spaced m grid points along x and y axes) and Ω_2 divided into n^2 (by taking uniformly spaced n grid points along x and y axes) rectangles. Fig. 4 illustrates an example with $m = 4$ and $n = 6$.

For $i = 1, 2$, let p_i, q_i, r be the polynomial degree of the approximation spaces S^i, N^i, T respectively. These degrees are chosen according to the necessary and sufficient conditions presented in Section 2.

First, we consider the h-version for the non-conforming method i.e. we fix the polynomial degree of the approximation spaces but keep refining the mesh to obtain a better error. Here we consider $(p_1, p_2, q_1, q_2, r) = (3, 3, 2, 2, 1)$ and the combinations $(m, n) \in \{(2, 3), (4, 6), \ldots, (10, 15)\}$. The results are shown in Table 1. We denote the number of *degrees of freedom* by DOF, which is the size of the matrix P in (2.22). The error is the percentage relative error that is calculated in a broken H^1 norm and it is clear from Table 1 that as we go from a course mesh ($(m, n) = (2, 3)$) to a really fine mesh ($(m, n) = (10, 15)$), the dimension of the problem (DOF) increases and the error progressively decreases. The convergence rate is observed to be of the order $O(h^{3/2})$.

Next, we consider the p-version for the non-conforming method i.e. we fix the mesh on each domain and keep refining the polynomial spaces to obtain a better error. Here we fix $(m, n) = (4, 6)$, $(q_1, q_2, r) = (2, 2, 1)$ and increase p_i from 3 to 7 for $i = 1, 2$. The results are shown in Table 2. Note that, the error decreases as we increase the polynomial degree of the spaces S^1 and S^2. It can also be seen that because of the low degree approximation of the spaces N^1, N^2, T, we do not see a drastic decrease in the error. Therefore we now consider the following experiment. We once again fix the mesh $(m, n) = (4, 6)$ as before, but now we consider $(p_1, p_2) = (4, 4)$ and the combinations $(q_1, q_2, r) \in \{(1, 1, 1),$

Table 1
The relative error in the energy norm (for fixed polynomial degree of spaces S^1, S^2, N^1, N^2 and T) in dependence on grid refinement

p_1	q_1	r	q_2	p_2	m	n	DOF	Error
3	2	1	2	3	2	3	111	7.042514
3	2	1	2	3	4	6	348	2.670658
3	2	1	2	3	6	9	715	1.480732
3	2	1	2	3	8	12	1212	0.969502
3	2	1	2	3	10	15	1839	0.696632

Table 2
The relative error in the energy norm (for fixed non-conforming mesh) in dependence on polynomial degree of spaces S^1, S^2

p_1	q_1	r	q_2	p_2	m	n	DOF	Error
3	2	1	2	3	4	6	348	2.670658
4	2	1	2	4	4	6	524	2.013736
5	2	1	2	5	4	6	752	1.897475
6	2	1	2	6	4	6	1032	1.859512
7	2	1	2	7	4	6	1364	1.854568

$(2,2,2),\ldots,(4,4,4)\}$. The results are shown in Table 3. Increasing the degree of the approximation spaces to the normal and traces automatically makes the error smaller as expected. Infact, since the exact solution chosen for our computations is of the order 4, our error exactly becomes zero for $(p_1,p_2,q_1,q_2,r)=(4,4,4,4,4)$.

Next, we simulate the conforming method by fixing the mesh to be $(m,n)=(4,4)$. So, now we have the meshes to be matching from both the domains on the interface AO. One must then be able to get good results by simply having good approximation for S^1 and S^2 even with a poor approximation for N^1, N^2, T. In order to test this we fixed $(q_1,q_2,r)=(1,1,1)$ and increased $(p_1,p_2)=\{(1,1),\ldots,(4,4)\}$ and the results are illustrated in Table 4. As predicted, the method now behaves as a conforming method and gives a zero error when $(p_1,p_2)=(4,4)$.

To conclude this section, we compare the exact point-wise function and derivative values extracted along the interface AO with those obtained via the non-conforming finite element method. For this, we extract 10 equally spaced

Table 3
The relative error in the energy norm (for fixed non-conforming mesh) in dependence on polynomial degree of spaces N^1, N^2 and T

p_1	q_1	r	q_2	p_2	m	n	DOF	Error
4	1	1	1	4	4	6	514	0.401059
4	2	2	2	4	4	6	528	0.117679
4	3	3	3	4	4	6	542	0.003988
4	4	4	4	4	4	6	556	0.000000

Table 4
The relative error in the energy norm (for fixed conforming mesh) in dependence on polynomial degree of spaces S^1, S^2

p_1	q_1	r	q_2	p_2	m	n	DOF	Error
1	1	1	1	1	4	4	64	19.470098
2	1	1	1	2	4	4	144	1.334426
3	1	1	1	3	4	4	224	0.074710
4	1	1	1	4	4	4	336	0.000000

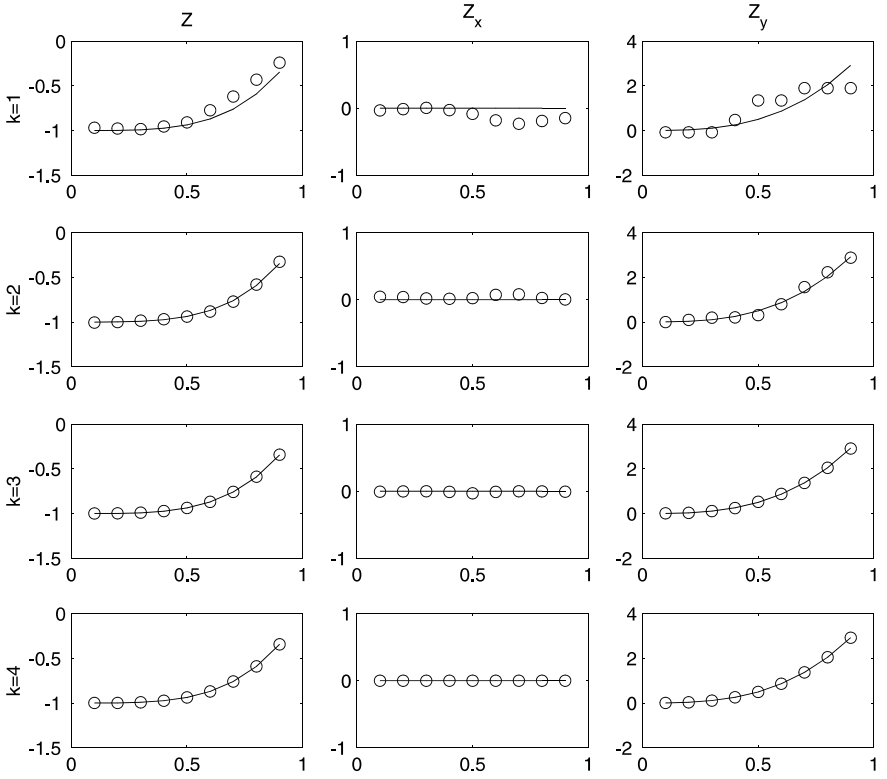

Fig. 5. Point-wise extraction of Z, Z_x and Z_y along AO.

points in [0.1, 0.9]. We fix the mesh to be $(m, n) = (2, 3)$, let the polynomial degrees p_i, q_i, r to be all the same $(= k)$ and consider the combinations $k = 1, \ldots, 4$. In Fig. 5, the values of Z, Z_x and Z_y as obtained from the exact solution are plotted by solid lines. Also plotted (with circles), are the values of Z, Z_x and Z_y obtained from the average of the non-conforming solutions (from Ω_1 and Ω_2). The figure clearly demonstrates that the approximation becomes better as k increases. We mention that even if the values of Z_x and Z_y are taken only from one side (without averaging), the results do not change much, and the accuracy remains the same. This suggests that stress extraction can be accurately performed even along interfaces when such non-conforming methods are used in elasticity problems. We also mention that the discontinuity in the solution across interfaces will be extremely small, as observed also in [12].

Remark 4.1. The computational results of this study demonstrate that the non-conforming technique presented, herein, is a good candidate for finite element

design of complicated domains. Despite being a two-dimensional study, it provides a good insight into much more complex problems. There is however, a need to generalize these to higher dimensions. It is clear from the formulation that one can employ this technique to obtain a better parallelization efficiency (the ratio of the computation time to communication time), although this needs to be tested. The latter two aspects will be the subject of a forthcoming paper.

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