DEVELOPING TEACHERS' REPRESENTATIONAL FLUENCY AND ALGEBRAIC CONNECTIONS

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This study examined the development of thirty-seven elementary and middle grades teachers' algebraic connections and representational fluency during a six-month professional learning project. To evaluate the collaborative nature of designing professional development, the team of professional developers/researchers used the collective self-study method (Samaras & Freese, 2006) to examine how purposively designed experiences such as the content-focused institute in the summer with school-based follow-up Lesson Study cycles (Lewis, 2002) in the fall encouraged vertical articulation of algebraic connections. The analysis of teachers' reflections from problem solving tasks and Lesson Study and researchers' memos from observed research lessons revealed more flexibility in teachers' representational fluency with problem solving strategies in the classrooms.

Introduction

Research and initiatives in elementary and middle school mathematics emphasize the importance of both fostering algebraic reasoning through problem solving and laying the critical foundations before students encounter formal algebra instruction (Blanton, 2008; Driscoll, 1999; NCTM, 2000; NMAP, 2008). This leads to some important practical questions: what experiences are required for elementary and middle grades teachers to effectively teach algebraic reasoning through problem solving, especially if they never learned mathematics in a reform oriented manner? What opportunities can professional development providers design for teachers to develop teachers' representational fluency? Reform oriented mathematics teaching presents challenges as teachers learn to change their practices (Remillard, 2000; Boaler, 1999). Such changes that advocate a reform-based vision of teaching algebra along with the great need for effective algebraic instruction in the K-12 curriculum has motivated mathematics educators, mathematicians and school leaders to emphasize more on studies about teachers' understanding of algebraic thinking and reasoning. Teaching algebraic reasoning in the earlier grades through problem solving requires depth of mathematical content knowledge for teaching. Hill, Sleep, Lewis and Ball (2007) suggest that specialized content knowledge is partially domain-specific and that the specified content knowledge for mathematics is not obtained based on an intelligence level, or ability to understand mathematics. Mathematics knowledge for teaching (Hill et. al, 2007) includes understanding of general content but also having domain specific knowledge of students and knowledge of content in algebra, patterns, and functions.

The purpose of the study was to provide a coherent picture of how teachers' algebraic connections and representational fluency develop through a content focused and practice-based professional development project. Our study set out to identify the key practice-based skills necessary for teachers to acquire the specialized knowledge for early algebra that would help them make algebraic connections to enhance their teaching and promote algebraic thinking in students.

Theoretic Framework

"Early algebra is not the same as algebra early. To move algebra-as-most-of-us-were-taught-it to elementary school is a recipe for disaster" (Carraher, Schelimann & Schwarts, 2008, p. 235).

The notion of teaching early algebra then is both a paradigm shift and a novel way of teaching that requires extensive professional development. According to Carraher et al. (2008), early algebra builds on background contexts of rich problems where early algebra tightly interweaves existing topics of early mathematics and formal notation is introduced gradually and "judiciously". Many researchers have explored teaching early algebra for the elementary grades and found that elementary students are capable of reasoning algebraically (Kaput, Carraher, & Blanton, 2008; Kaput, 2008; Bastable & Schifler, 2007; Kieren, 2004; Carpenter & Levi, 2000; Greenes & Findell, 1999; Moses, 1993). This integrative approach of algebraic reasoning and arithmetic connections is common in many international elementary and middle school mathematics curriculum (e.g., China, Russia, Singapore, and South Korea), where students begin the formal study of algebra much earlier (Cai, 2004). Such integrative approaches have helped these students in building mathematical understanding with skill development, in embedding the arithmetic in interesting and challenging problem situations, in enhancing their number sense through purposeful calculations, and in relating problems to one another and to other essential mathematics content knowledge they are building constantly.

According to the NCTM (2000) the underlying concepts of algebra are patterns, relations, and functions and the way in which they are utilized to: 1) represent and analyze mathematical situations and structures using algebraic symbols; 2) represent and understand quantitative relationships; and 3) analyze change in various contexts (NCTM, 2000, p. 39). Blanton and Kaput's (2008) Generalizing to Extend Arithmetic to Algebraic Reasoning (GEAAR), project goal was to embed algebraic thinking into instruction and to build teacher capacity to "algebrafy their classrooms" (p. 384). The evidence to support their findings was gathered from teachers' reflections and written work, students' reflections and written work, observations, and interviews. This professional development addressed both teacher content knowledge as well as classroom practices. Blanton and Kaput (2008) reported that teachers become better at teaching algebraic reasoning when the teachers' own mathematical knowledge and understanding are increased. According to Blanton and Kaput (2008), teachers develop their algebra "eyes and ears" which allow them to bring out the algebraic reasoning while looking at student work and carefully listening to their discussions and questions. This requires teachers having an integrated depth of algebra content knowledge to know what to look and listen for in the classroom. Developing algebraic habits of minds (Driscoll,1999) include the idea of seeing algebra as: 1) doing and undoing; 2) building rules to represent a function; and 3) abstracting from computation. Developing algebraic habits of mind is critically important to the success of our elementary, middle and high school teachers. It takes a teacher who has a deep and profound understanding of fundamental algebra to provide opportunities for elementary and middle grades students to explore the foundation concepts for algebraic reasoning through patterning, relations, functions, and representations using algebraic symbols and utilizing mathematical models to represent relationships (NCTM, 2000).

For our study, we defined 'algebraic connection' as the integrated knowledge developed by applying algebraic concepts to interdisciplinary problem-solving opportunities through multiple representations and flexible strategies. Such connections help to discover and comprehend the applications of mathematical concepts to real-world problems in a way that is beyond what is taught in a traditional classroom. Thus, 'algebraic connection for teachers' included some very specific practiced-based skills (such as, making: 1) cross-curricular (horizontal) connections- algebraic connections to real world situations and interdisciplinary problem-solving; 2) 'algebra-arithmetic' (vertical) connections- connecting arithmetic structures such as addition, subtraction, multiplication, division, ratios, proportional thinking, rational numbers and geometry to algebra; making generalizations about a concept; 3) algebraic connections through representations orchestrating mathematics discussions through the use of students' physical, tabular graphical models, verbal and symbolic notations to build representational fluency.

One of the key performance skills was for teachers to teach algebraic lessons using the five star representations for all algebra lessons. The five star representation connected representational systems such as using pictures, number lines, graphs, tables, arithmetic notations and verbal description. (Figure 1) Representational fluency, the ability to use multiple representations and translate among these models, has been shown to be critical in building students' mathematical understanding (Goldin & Shteingold, 2001).

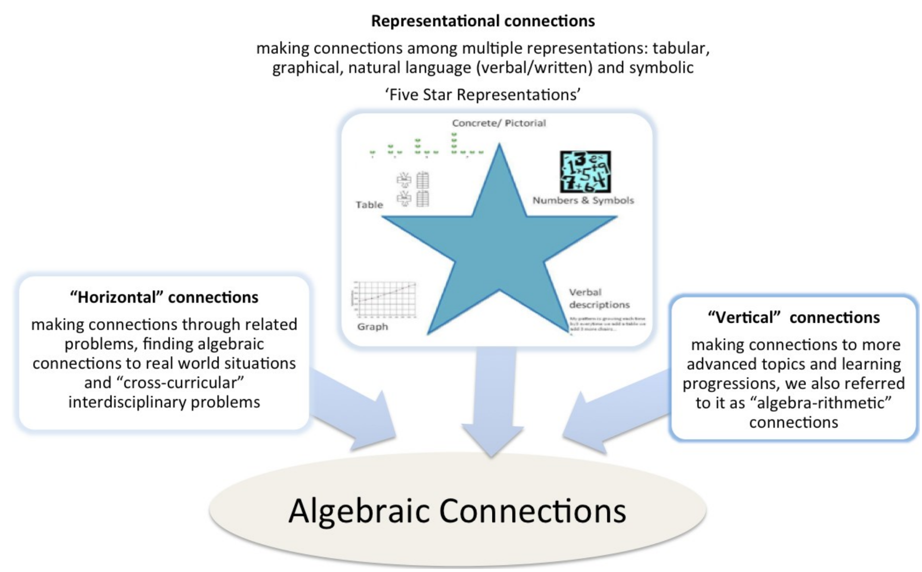

Figure 1

Methods

Research Questions

The goal of the study was to understand the processes of change in participants' teaching practices and examining what professional development activities and events were pivotal in influencing that change. This study explored the following two research questions: 1) what changes in teachers' development of the algebraic connections were evident through the activities and professional learning opportunities and 2) what are some essential professional development opportunities and support necessary for teachers as they transition into teaching through problem solving and making algebraic connections?

Data Sources

This study examined teachers' reflections and field notes from two phases of the professional development project: the summer institute where teachers were immersed in algebra problem solving activities as learners and the second phase with the follow-up Lesson Study where teachers implemented teaching practices that promoted algebraic connections through rich problem solving.

Participants

Thirty-seven elementary and middle grades teachers from grades 3 - 8 met for a one week summer institute and continued as school teams in Lesson Study (Lewis, 2002) during the academic year. A majority of the teachers (78%) taught in Title One schools that served underrepresented and underserved populations. On average, teachers had 12 years of experience, where 18 held baccalaureate degrees and 19 held Master degrees. Six teachers were mathematics specialists, and three teachers were special educators.

Procedures

To evaluate the collaborative nature of designing professional development, the team of professional developers/researchers used the collective self-study method (Samaras & Freese, 2006) to examine how purposively designed experiences such as the content-focused institute in the summer with school-based follow-up Lesson Study cycles in the fall encouraged vertical articulation of algebraic connections. The method for analysis involved a mixed method approach of survey analysis and qualitative analysis. Using the Grounded Theory approach (Strauss & Corbin, 1994), we used a constant comparative method. This method allowed for us to systematically gather and analyze data through the summer institute and Lesson Study cycle and to generate a theory that was grounded in these data. The researchers used Self-Study to determine what designed activities promoted the development of representational fluency. During the summer and follow-up Lesson Study session, we debriefed on the project activities and analyzed the teachers' reflections from the activities. Upon analysis, we searched for recurring themes and categories that linked various data. The analysis of the data from teacher reflection began with the reading of the reflections from the summer institute during which patterns in the participants responses were identified. The data analysis was aimed at answering the research questions and identifying themes, categories, or types (Miles & Huberman, 1994). Successive readings of the data necessitated modification and further development of the coding categories.

Results

Opportunities to Deepen Teacher Mathematical Knowledge and Algebraic Connections

To address research questions, "what changes in teachers' development of the algebraic connections were evident through the activities and professional learning opportunities", and "what essential professional development opportunities and support are necessary for teachers as they transition into teaching through problem solving and making algebraic connections?" we analyzed the reflections from problem solving tasks during the content-focused summer institute and from the Lesson Study.

The analysis of teachers' reflections revealed four main themes: teachers developed a) an awareness of their own metacognitive process during problem solving tasks; b) an appreciation of the need for specialized mathematics knowledge; c) connections to teachers' classroom practices based on one's own reflection from tasks; d) awareness of the affective nature revealed during the learning processes. Below are representative excerpts from the qualitative analysis.

Awareness of teacher's own metacognitive process during problem solving tasks

The developing awareness of one's own metacognitive process encouraged reflection upon personal learning and problem solving style preferences, identification of common pitfalls and misconceptions that accompanied the problem solving process, and the ability to internally perform task analysis in order to generate multiple problem solving ideas.

After Dr. S. shared how Gauss' thinking connected to this problem, it seems to keep coming up in other problems we have solved. It all started with the handshake problem and then it appeared again in the toothpick problem, the fruit stack problem, and the penny jar problem. While I could identify and extend a triangular number pattern, the importance of the pattern was not something I thought was significant. I don't ever recall noticing this triangular number pattern, and it seems to keep coming up in numerous problems. 1, 3, 6, 10…now, I keep seeing this pattern everywhere. – Elementary math specialist

Appreciation of the need for specialized mathematics knowledge

Participants noted that using multiple strategies and representations to solve rich questions that required deeper mathematical thinking increased their understanding of algebra concepts. Teacher learners immersed in a content-focused institute and asked to explore novel approaches and unfamiliar tools and technology experienced an expanded appreciation of algebraic connections and representational fluency.

I noticed that when he explained his strategy to our other partners, his visual really seemed to be clearer than my abstract expression. That was very powerful for me because I need to always keep in mind as a teacher that all of my students learn in different ways and are all at very different levels. While some of my students may be ready for the algebraic expression, there are so many important benefits that can be gathered from the pictorial/graphical representations as well. -7th grade teacher

Making connections to teachers' classroom practices based on one's own reflection from tasks

Throughout the institute, participants grappled with problems as students themselves and saw the value of rich problems that encourage multiple approaches and rich discussion. They personally experienced the connections that exist algebraically within and between the mathematical strands of number, geometry, measurement, and statistics. Scaffolding opportunities, differentiation potential and readiness assessment were all named as benefits of this type of learning environment. However, many did voice concern over barriers within their schools that could impede implementing these newly learned and appreciated reform practices.

Obviously I have thought about how I might use this problem in my own classroom this year. I am excited to see what algebra I can bring out for my students and how I can modify a lesson focused on inequalities to a new lesson focused on algebraic expressions.

As I think through how I will adjust and focus this lesson for my sixth grade students I am oriented on two main factors. First, I need to consider multiple representations and access points in my classroom. I need students to feel comfortable seeing problems in multiple representations and then get them comfortable trying to use these approaches (concrete, pictorial, graph, symbolic, verbal) to solve problems. I also must consider how I present a problem—how can I make sure all of my students have an access point to solving a problem?- 6th grade teacher

Awareness of the affective nature revealed during the learning processes.

The teacher learners voiced both struggles and satisfaction during the problem solving processes. They appreciated the collaboration and support from colleagues and emphasized the importance of mathematics communication to building mathematical ideas.

The next two classes were painful to say the least, by Thursday I was ready to tell my partner I couldn't do this. She talked to me all the way to class but I was not really convinced. Then like a flash, all start falling into place. As I pass the pages, I look at the pizza problem. That was the one that gave me confidence. I saw the connection with the handshake problem. I thought, what was different this time. As I was thinking about that, Habits of Mind came to *my thoughts. I realized that I was beginning to use them in my problem solving. That is something I want to impart in my students! – 4th grade teacher*

Catalysts for Change: Essential Opportunities for Teachers to Develop Algebraic Connections

As we coded teachers' reflections from the designed project activities, key events stood out as being pivotal points in eliciting teachers to rethink their practices. We identified these key activities as the Catalysts for Change in our teachers because these activities seemed to be the events in which teachers realized they needed to change their way of thinking about their instructional practices for mathematics teaching and learning.

Key Activity 1: Teachers immerse in rich problems with algebraic connections

It was important for teachers to grapple with the problems and experience disequilibrium. This opportunity allowed for teachers to make connections to the fundamental algebra and increase their understanding of the topic. It was a "relearning" process for many who were making sense of the algebra that they had learned procedurally; void of context or real life applications. These connections were made by both experienced and novice teachers. We coded this as '*making connections and see algebra in a new way'*. An experienced eighth grade teacher who currently taught Algebra was able to see the content in a new way. One excerpt from her reflection said,

> *Just as I needed to make connections between the context, my rule, and why it worked, students of all ages need these same opportunities. Previously when teaching students about triangular numbers, I focused on the number pattern and the geometric pattern. It was solely identifying what is happening by doing and undoing and extending the pattern, perhaps even getting to build a rule. What was totally missing from my instruction? Problems in which triangular number patterns actually appeared! Pretty big I think. Just as my understanding of triangular numbers and algebra deepened, students will receive the same benefit by having opportunities to take their understanding of patterns and apply it to more complex situations.* -8th grade teacher

We also noticed that these activities elicited teachers developing representational fluency and valuing the use of multiple representations. A novice 5th grade teacher excerpt was coded *'developing representational fluency'* because she expressed using multiple representations as she solved and made sense of the problem and also made connections to how her students might gain access to the problem through a visual approach.

> *Reflecting back on how I solved the problem I notice that I used many different representations to help me solve the problem. First I drew a picture, then I used a table, and last I used symbols. This idea of multiple representations was accented in class and made me process how students might need different entry points to be able to engage in this problem. It made me think about how to set up a problem and get students started on a problem. The visual used in class was extremely helpful for me, even though I could access the problem in other ways. It reinforced what I saw was going on in the problem, but had I been a child that could not see what was going on, it would have given me an opportunity to engage in the problem (*5th grade math and science teacher).

Key activity 2: Opportunities for 'teacher math talk' on algebra and vertical articulation

The importance of collaboration and vertical articulation came to the forefront as teachers discussed problems with their colleagues and talked about the grade level mathematics expectations. One teachers commented how her team members helped her build on her existing ideas about algebra.

As we have worked through problems this week, I have been very fortunate to sit with colleagues who teach at a range of grade levels. Our table actually represented grades 3-6. As we shared our thinking and problem solving strategies, I learned to see the big, intricate picture from them. There were times when it seemed that they gained a lens by hearing about my analysis of a table, my exploring with manipulatives, or through my illustrations. -3rd grade teacher

This was a great connection to what research has noted about how students learn from class discussion (Hufferd-Ackles, Fuson, & Sherin, 2004). We observed teachers learning from the mathematical discussions, questioning, explaining mathematical thinking, and debating mathematical ideas. This critical activity also elicited teachers' responses to recognizing the importance of breaking down the essential mathematical learning, identifying the common student misconceptions and scaffolding and differentiating for diverse learners.

I feel most prepared to teach my students to use multiple representations, to notice patterns, and build rules. I think it was most helpful to solve problems collaboratively in the summer class and to share our solutions. It was interesting to see the various ways my classmates solved the problems. -Special educator who teaches in an inclusive classroom

In many ways, this opportunity allowed teachers to simulate the math talk that they would conduct in their own classroom. Having a chance to discuss potential misconception, analyze sophistication of solution strategies, and ways to create parallel tasks for differentiation allowed teachers to have a natural conversation of progression of mathematical ideas and vertical articulation.

Discussion

At the end of the course, participants reflected on how their participation in the yearlong professional development session had impacted their ideas of algebra in the early grades. The most common themes were that they "relearned" the math by being in the "shoes of the student" and having to solve challenging problems while breaking down the important mathematics helped them see the early building blocks for creating algebraic connections. These experiences not only increased their confidence in the teaching and learning of mathematics but also helped the participants to develop a productive disposition towards mathematics.

This study revealed that engaging teachers in challenging activities that created some sense of disequilibrium led teachers to reflect on their knowledge, their problem solving process, and their instructional practice. Participants' positive experiences during the learning process helped develop a more productive disposition towards teaching algebraic reasoning through problem solving and through multiple representations, and value collaborating with other mathematics educators in translating their learning into practice. In addition, the study demonstrated the effectiveness of the choice of rich problems that helped us to study teacher's developing knowledge and that also provided the teachers with useful examples for algebraic instruction and student learning assessment. It is the authors' hope that the results of this study contribute to the growing body of knowledge documenting how professional development serves as a catalyst for change in teachers as they take time to reflect and relearn specialized knowledge for teaching algebra in the early to middle grades.

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