Numerical Analysis **Homework 2**
Math-685/OR682 **Deadline: Oct 23, 2017**

1. (10 points) The speed of wave propogation of a surface wave in a fluid flowing in a channel is given by,
$$v = \sqrt{\frac{g\lambda}{2\pi}\ \tanh\left(\frac{2\pi y}{\lambda}\right)}$$
where v is the speed of the wave, λ is the length of the wave, y is the depth of the fluid and g is the acceleration due to gravity. Use the bisection method to find the value of $\frac{\lambda}{y}$ to within 10^{-4} for which
$$\frac{v}{\sqrt{gy}} = 0.99.$$

2. (10 points) Consider the sequence $\{x_k\}_{k=0}^{\infty}$ defined by
$$x_{k+1} = g(x_k) = \gamma f(x_k) + h(x_k)$$
for $k = 0, 1, 2, \ldots$ where $\gamma > 0$, x_0 is a real number, f, h are real valued functions, and $|h'(x)| \leq \frac{3}{4}$. Find a condition on γ that will guarantee convergence of the sequence $\{x_k\}_{k=0}^{\infty}$ to the unique real number z such that $z = g(z)$.

3. (15 points) Consider the function $g(x) = e^{-x}$ on the interval $G = [\ln 1.1, \ln 3]$
   (a) Show that $g(x)$ has an unique fixed point that is real number.
   (b) Prove that $g(x)$ is a contraction on G.
   (c) Prove that $g(x)$ maps the interval G to itself.

4. (15 points) Consider the iteration function $g(x)$ of the form $g(x) = x - f(x)f'(x)$, where $f(r) = 0$ and $f'(r) \neq 0$. Find the precise conditions on the function f so that the iterations $x_{n+1} = g(x_n)$ converge to the fixed-point r at least cubically if started near r.

5. (15 points) Consider the Newton-Raphson iterative method.
   (a) Establish the iterative sheme, $x_{n+1} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right)$, to calculate the cube root of a number N.
   (b) Use the relationship in part (a) to perform two iterations of the Newton-Raphson scheme to obtain an approximation to $\sqrt[3]{12}$.

6. (15 points) Consider solving the system of nonlinear equations:
$$\begin{aligned} xy &= z^2 + 1 \\ xyz + y^2 &= x^2 + 2 \\ e^x + z &= e^y + 3 \end{aligned}$$
Starting with $(1, 1, 1)^T$, perform three iterations of the Newton-Raphson method to determine the approximate roots of the system of equations.

7. (15 points) A can in the shape of a right circular cylinder is to be constructed to contain 1000 cm^3. The circular top and bottom of the can must have a radius of 0.25 cm more than the radius of the can so that the excess can be used to form a seal with the side. The sheet of material being formed into the side of the can must also be 0.25 cm longer than the circumference of the can so that a seal can be formed. Find, to within 10^{-4}, the minimal amount of material needed to construct the can.

8. (25 points) Consider the following method for evaluating the smallest root of an equation $f(x) = 0$ where $f(x) = 1 - \left(a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots\right)$. For smaller values of x we can write (using binomial theorem):

$$\left[1 - \left(a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots\right)\right]^{-1} = b_1 + b_2x + b_3x^2 + \ldots$$

where we have:

$$\begin{aligned} b_1 &= 1 \\ b_2 &= a_1 \\ b_3 &= a_1^2 + a_2 = a_1b_2 + a_2b_1 \\ \ldots & \\ \ldots & \\ b_n &= a_1b_{n-1} + a_2b_{n-2} + \ldots + a_{n-1}b_1. \end{aligned}$$

Then $\lim_{n\to\infty} \frac{b_n}{b_{n+1}}$ converges to a root of the equation $f(x) = 0$.

(a) Write a MATLAB program to implement the method outlined above.

(b) Use the program in part (a) to find the smallest root of the equations within a tolerance of 10^{-4}.

i. $x^3 - 5x^2 + 12x - 8 = 0$.

ii. $x + \sin x - 1 = 0$