Numerical Analysis **Homework 1**
Math-685/OR682 **Deadline: Sept 25, 2017**

1. (10 points) Suppose $f \in C[a,b]$, that $x_1 \le x_2 \le x_3$ are in $[a,b]$. Show that there exists a number ξ between x_1 and x_3 with,

$$f(\xi) = \frac{f(x_1) + 2f(x_2) + 3f(x_3)}{6}.$$

2. (10 points) Suppose function f has a continuous fourth derivative. Show that:

$$\left| \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - f''(x) \right| \le ch^2.$$

3. (10 points) The function $f(x) = e^{x/2}$ is to be evaluated for any x, $(0 \le x \le 25)$ correct to 5 significant decimal digits. What digit decimal rounding arithmetic should be used (i.e., in x) to get the required accuracy in evaluating the function $f(x)$?

4. (10 points) Suppose that $fl(y)$ is a k-digit rounding approximation to y. Show that:

$$\left| \frac{y - fl(y)}{y} \right| \le 0.5 \times 10^{-k+1}$$

5. (20 points) Consider the *error function* defined by:

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

that arises in several real-world applications that involves the integrals of gaussian distributions. While this integral cannot be evaluated in terms of elementary functions, the following approximating technique may however be used.

(a) Using a Taylor Series expansion about $x = 0$, derive the following:

$$erf(x) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)k!}$$

(b) Write a MATLAB program to approximate $erf(x)$ by the above approximations for $k = 2, 5, 10$. Compare the $erf(x)$ function from MATLAB with the approximate functions implemented for $k = 2, 5, 10$ respectively by plotting over the range $[-3, 3]$.

6. (20 points) Consider the function $f(x) = \dfrac{e^x - e^{-x}}{x}$.

   (a) Use three-digit rounding arithmetic to evaluate $f(0.1)$.

   (b) The actual value of $f(0.1) = 2.003335000$. Find the relative error obtained by using the value obtained in part (a).

   (c) Use an efficient method to evaluate the function $f(x)$ for small values of x. Find the relative error obtained by using this method for evaluating $f(0.1)$.

7. (20 points) Let $f(x) = 1.01e^{4x} - 4.62e^{3x} - 3.11e^{2x} + 12.2e^x - 1.99$.

   (a) Use three-digit rounding arithmetic, the assumption that $e^{1.53} = 4.62$ and the fact that $e^{nx} = (e^x)^n$ to evaluate $f(1.53)$.

   (b) Repeat part(a) after rewritting $f(x)$ in a nested fashion.

   (c) Compare the approximations in parts (a) and (b) to the true three-digit result $f(1.53) = -7.61$.

8. (20 points) Let x_i^* for $i = 1, 2, 3, 4$ be positive numbers. With a unit round-off error δ, $x_i^* = x_i\ (1 + \epsilon_i)$ with $|\epsilon_i| \leq \delta$, where x_i for $i = 1, 2, 3, 4$ are the *exact* numbers. Consider the scalar dot product

$$S_2 = \vec{a} \cdot \vec{b} = x_1 x_3 + x_2 x_4$$

of two vectors $\vec{a} = [x_1, x_2]$ and $\vec{b} = [x_3, x_4]^T$. Let S_2^* be the *floating point approximation* of S_2. Prove that

$$\frac{S_2^*}{S_2} \leq e^{4\delta}.$$