BASIC THEORY OF FINITE ELEMENT METHOD

Consider the Boundary Value Problem:

$$-u''(x) + u(x) = f(x)$$

$$u(0) = 0$$

$$u(1) = 0$$

Let us define the space V as follows:

$$V = \left\{ u \ni: \int_0^1 (u'(x))^2 + (u(x))^2 < \alpha \text{ and } u(0) = u(1) = 0 \right\}$$

Let us also define the norm on the space to be

$$\|u\|_V = \left(\int_0^1 (u'(x))^2 + (u(x))^2 \right)^{1/2}$$

Note that the weak-variational problem for the BVP becomes: Find $u(x) \in V \ni$:

$$\int_0^1 u'(x)\, v'(x)\, dx + \int_0^1 u(x)\, v(x)\, dx = \int_0^1 f(x)\, v(x)\, dx \quad \forall v(x) \in V$$

Let us define a bilinear functional

$$a(u,v) := \int_0^1 u'(x)\, v'(x)\, dx + \int_0^1 u(x)\, v(x)\, dx$$

and a linear functional:

$$F(v) := \int_0^1 f(x)\, v(x)\, dx$$

Our problem then becomes: find $u(x) \in V$ that satisfies $a(u,v) = F(v) \quad \forall\ v(x) \in V$.

For this problem to have an unique solution we have the following theorem.

LAX-MILGRAM THEOREM

(A) Let V be a Hilbert space with scalar product $(\cdot,\cdot)_V$ and norm $\|\cdot\|_V = (\cdot,\cdot)_V^{1/2}$

(B) Let $a(\cdot,\cdot)$ be a symmetric bilinear form on $V \times V$ such that:

(i) $a(\cdot,\cdot)$ is Continuous (bounded) that is

$\exists$ a Constant $\gamma > 0 \ni$:

$$|a(v,w)| \leq \gamma \|v\|_V \|w\|_V \quad \forall v,w \in V$$

(ii) $a(\cdot,\cdot)$ is V-elliptic (Co-ercive) that is

$\exists \ \alpha > 0 \ni$:

$$a(v,v) \geq \alpha \|v\|_V^2 \quad \forall v \in V$$

(c) Finally, suppose that there is a Continuous (bounded) linear form F on V i.e.

$\exists$ a Constant $\lambda > 0 \ni$:

$$|F(v)| \leq \lambda \|v\|_V \quad \forall v \in V$$

Then $\exists$ a unique solution $u(x) \in V$ that satisfies:

$$a(u,v) = F(v) \quad \forall v \in V$$

For the BVP considered note that,

$$|a(v,w)| = \left| \int_0^1 v'(x)\, w'(x)\,dx + \int_0^1 v(x)\, w(x)\, dx \right|$$

TRIANGLE INEQUALITY

$$\leq \left| \int_0^1 v'(x) w'(x)\, dx \right| + \left| \int_0^1 v(x) w(x)\, dx \right|$$

CAUCHY SCHWARZ

$$\leq \left(\int_0^1 (v'(x))^2 dx \right)^{1/2} \left(\int_0^1 (w'(x))^2 dx \right) + \left(\int_0^1 (v(x))^2 dx \right)^{1/2} \left(\int_0^1 (w(x))^2 dx \right)^{1/2}$$

$$\leq 2 \left(\int_0^1 (v'(x))^2 + (v(x))^2 \right)^{1/2} \left(\int_0^1 (w'(x))^2 + (w(x))^2 \right)^{1/2}$$

$$= 2 \|v\|_V \|w\|_V$$

Hence we have shown that $a(v,w)$ is bounded.

Also $$a(v,v) = \int_0^1 \left((v'(x))^2 + (v(x))^2 \right) dx$$

$$\geq C \int_0^1 \left((v'(x))^2 + (v(x))^2 \right) dx \quad \text{With } C = 1$$

$$= C \|v\|_V \quad \text{((COERCIVITY)}$$

Finally we also note that

$$\left| \int_0^1 f(x)\, v(x)\, dx \right| \leq \left(\int_0^1 (f(x))^2 dx \right)^{1/2} \left(\int_0^1 (v(x))^2 dx \right)^{1/2}$$

$$\leq \|f\|_2 \; \|v\|_V$$

Hence choosing $\lambda = \|f\|_2$ we also note that $F(v)$ is bounded. Therefore we can now employ the Lax-Milgram theorem which yields the result that the solution to the BVP given is unique.

To detemine an approximate solution, we consider a finite dimensional space $V_N \subset V$ and we look for $u_h(x) \in V_N$, which also satisfies:

$$a(u_h, v) = F(v) \quad \forall v \in V_h$$

Since the exact solution satisfies $(u(x) \in V)$

$$a(u, v) = F(v) \quad \forall v \in V_h$$

Subtracting these equations we get,

$$a(u - u_h, v) = 0 \quad \forall v \in V_h$$

which is an important orthogonality relationship satisfied by the approximate solution $u_h(x)$. Let us now try to use this orthogonality relation to prove the following basic error estimate.

THEOREM: $\|u-u_h\| \leq \dfrac{\gamma}{\alpha}\|u-w_h\| \quad \forall w_h \in V_N$

Let $w_h \in V_N$, $v_h \in V_N$ such that $v_h = u_h - w_h$.
From coercivity we have,

$$\alpha \|u-u_h\|^2 \leq a(u-u_h,\ u-u_h)$$

$$= a(u-u_h, u-u_h) + a(u-u_h, u_h-w_h)$$

USING ORTHOGONALITY

$$= a(u-u_h,\ u-w_h)$$

BOUNDEDNESS

$$\leq \gamma \|u-u_h\|\,\|u-w_h\|$$

$$\Rightarrow \|u-u_h\| \leq \frac{\gamma}{\alpha}\|u-w_h\| \quad \forall w_h \in V_N$$