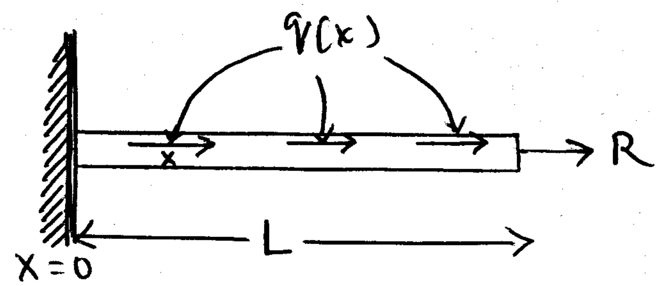
# Introduction to Finite Element Methods
(Padmanabhan Seshaiyer)

Let us consider the simplest solid mechanics problem : AXIAL LOADING OF A BAR




$q(x)$ : DISTRIBUTED AXIAL LOAD (Force per unit Length)

$R$ : CONCENTRATED AXIAL LOAD AT $X=L$ (Force)

$L$ : LENGTH OF THE BAR

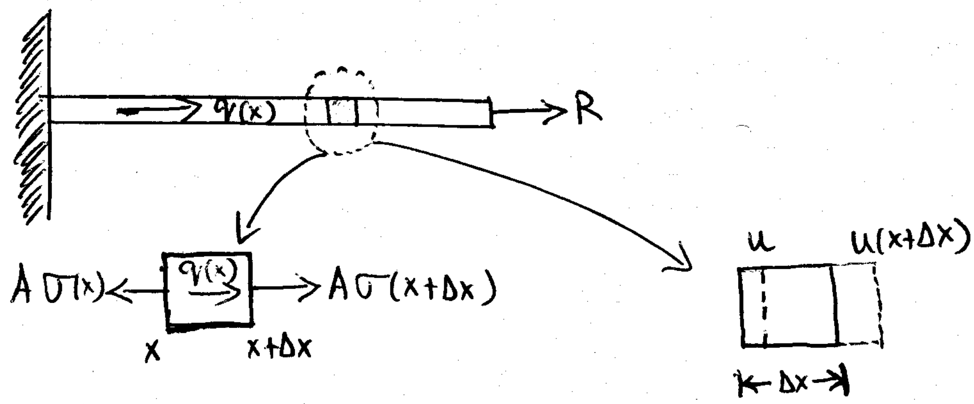
$A$ : AREA OF CROSS-SECTION

$E$ : YOUNG'S MODULUS OF MATERIAL

## ASSUMPTIONS:

(*) The cross-section of the bar does not change after loading.

(*) The material is linear elastic, isotropic and homogeneous.




$\sigma(x)$ : Stress (= Force per unit area)

$\varepsilon(x)$ : Strain (= Change in length/original length)

**FORCE BALANCE:** $A\sigma(x) = q(x)\,\Delta x + A\sigma(x+\Delta x)$

$$\Rightarrow \quad A\,\frac{\sigma(x+\Delta x)-\sigma(x)}{\Delta x} + q(x) = 0$$

Passing the limit as $\Delta x \to 0$ we get

$$A \lim_{\Delta x \to 0} \frac{\sigma(x+\Delta x)-\sigma(x)}{\Delta x} + q(x) = 0$$

$$\Rightarrow \quad A\,\frac{d\sigma}{dx} + q(x) = 0 \quad \text{— (1)}$$

**CONSTITUTIVE EQUATION:** $\sigma(x) \propto \varepsilon(x)$ (Hooke's Law)

$$\Rightarrow \quad \sigma(x) = E\,\varepsilon(x) \quad \text{— (2)}$$

## STRAIN-DISPLACEMENT RELATIONSHIP:

$$\varepsilon := \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} \Rightarrow \varepsilon(x) = \frac{du}{dx} \quad \text{—③}$$

From equations ①, ② and ③ we get,

$$AE \frac{d^2u}{dx^2} + q(x) = 0$$

## BOUNDARY CONDITIONS:

At the wall ($x=0$) the displacement is zero.

$$\Rightarrow \quad u(0) = 0$$

At the free end ($x=L$) the force is prescribed as $R$. Since the force at the free end is $A\sigma(L)$ we have:

$$A\,\sigma(L) = R$$

From ② and ③ we then have

$$AE \frac{du}{dx}(L) = R$$

The model problem then becomes: Find $u(x) \ni$:

$$\boxed{\begin{aligned} AE\frac{d^2u}{dx^2} &= -q(x) \qquad 0 < x < 1 \\ u(0) &= 0 \\ \frac{du}{dx}(L) &= \frac{R}{AE} \end{aligned}} \quad \text{— (I)}$$

HOMEWORK: Let $q(x) = -ax$

Show that the solution to (I) is given by

$$u(x) = \frac{-ax^3 + (6R + 3aL^2)x}{6AE}$$

and $$\sigma(x) = \frac{-ax^2 + (2R + aL^2)}{2A}$$

Also plot the exact solution $u(x)$ and stress $\sigma(x)$.

## WEAK VARIATIONAL FORM:

**Step 1:** Multiply the differential equation in (I) by a test function $v(x)$ that satisfies $v(0) = 0$. We have,

$$AE \frac{d^2 u(x)}{dx^2} v(x) = -q(x)\, v(x)$$

**Step 2:** Integrate the equation in Step 1 by parts to yield:

$$\int_0^L AE \frac{d^2 u}{dx^2} v(x)\, dx = -\int_0^L q(x)\, v(x)\, dx$$

$$\Rightarrow AE\, v(x) \frac{du(x)}{dx} \Big|_0^L - \int_0^L AE \frac{du}{dx}\frac{dv}{dx}\, dx = -\int_0^L q(x)\, v(x)\, dx$$

**Step 3:** Apply the boundary conditions $v(0) = 0$ to yield

$$AE\, v(L) \frac{du}{dx}(L) - \int_0^L AE \frac{du}{dx}\frac{dv}{dx}\, dx = -\int_0^L q(x)\, v(x)\, dx$$

Since $\frac{du}{dx}(L) = \frac{R}{AE}$ we then have,

$$-\int_0^L AE \frac{du}{dx}\frac{dv}{dx}\,dx = -\int_0^L q(x)\,v(x)\,dx - AE\,v(L)\cdot\frac{R}{AE}$$

Integral on weak variational Form

$$\Rightarrow \boxed{\int_0^L AE \frac{du}{dx}\frac{dv}{dx}\,dx = \int_0^L q(x)\,v(x)\,dx + R\,v(L)}$$

## FINITE - DIMENSIONAL APPROXIMATION :

We now look for an approximate solution $u_h(x)$ in a finite dimensional space such that

$$\int_0^L AE \frac{du_h}{dx}\frac{dv}{dx}\,dx = \int_0^L q(x)\,v(x)\,dx + R\,v(L)$$

for all $v(x)$ from the finite dimensional space.

Let the finite dimensional space be spanned by a basis $\{\phi_1(x), \phi_2(x), \ldots \phi_N(x)\}$

Then we can write $u_h(x)$ as a linear combination as

$$u_h(x) = \sum_{i=1}^{N} c_i \, \phi_i(x)$$

$$= c_1 \phi_1(x) + c_2 \phi_2(x) + \cdots c_N \phi_N(x)$$

where $c_1, c_2 \cdots c_N$ are unknown coefficients. Also selecting $v(x)$ to be any of the basis functions $\phi_j(x)$ $j = 1, 2 \cdots N$, we then have,

$$\int_0^L AE \frac{d}{dx}\left(\sum_{i=1}^{N} c_i \phi_i(x)\right) \frac{d}{dx} \phi_j(x)\,dx = \int_0^L q(x)\,\phi_j(x)\,dx + R\,\phi_j(L)$$

$$j = 1, 2 \cdots N$$

$$\Rightarrow \int_0^L AE \left(\sum_{i=1}^{N} c_i \frac{d\phi_i}{dx} \frac{d\phi_j}{dx}\right) dx = \int_0^L q(x)\,\phi_j(x)\,dx + R\,\phi_j(L)$$

$$j = 1, 2 \cdots N$$

Writing out this for each $j = 1, 2 \cdots N$ yields a system of equations.

$$\int_0^L AE\left(c_1\frac{d\phi_1}{dx}\frac{d\phi_1}{dx}+c_2\frac{d\phi_2}{dx}\frac{d\phi_1}{dx}+\cdots\cdots c_N\frac{d\phi_N}{dx}\frac{d\phi_1}{dx}\right)dx=\int_0^L q(x)\,\phi_1(x)\,dx+R\,\phi_1(x)$$

$$\int_0^L AE\left(c_1\frac{d\phi_1}{dx}\frac{d\phi_2}{dx}+c_2\frac{d\phi_2}{dx}\frac{d\phi_2}{dx}+\cdots\cdots c_N\frac{d\phi_N}{dx}\frac{d\phi_2}{dx}\right)dx=\int_0^L q(x)\,\phi_2(x)\,dx+R\,\phi_2(x)$$

$$\vdots$$

$$\int_0^L AE\left(c_1\frac{d\phi_1}{dx}\frac{d\phi_N}{dx}+c_2\frac{d\phi_2}{dx}\frac{d\phi_N}{dx}+\cdots c_N\frac{d\phi_N}{dx}\frac{d\phi_N}{dx}\right)dx=\int_0^L q(x)\,\phi_N(x)\,dx+R\,\phi_N(x)$$

which is a system of equations in integral form. This system can be written in a Matrix form

$$\boxed{K\vec{c}=\vec{F}}$$

where, $K$ = Stiffness matrix & $\vec{F}$ = Load vector

$$K = AE \begin{bmatrix} \int_0^L \frac{d\phi_1}{dx}\frac{d\phi_1}{dx} & \int_0^L \frac{d\phi_2}{dx}\frac{d\phi_1}{dx} & \cdots\cdots & \int_0^L \frac{d\phi_N}{dx}\frac{d\phi_1}{dx} \\ \int_0^L \frac{d\phi_1}{dx}\frac{d\phi_2}{dx} & \int_0^L \frac{d\phi_2}{dx}\frac{d\phi_2}{dx} & \cdots\cdots & \int_0^L \frac{d\phi_N}{dx}\frac{d\phi_2}{dx} \\ \vdots & & & \\ \int_0^L \frac{d\phi_1}{dx}\frac{d\phi_N}{dx} & \int_0^L \frac{d\phi_2}{dx}\frac{d\phi_N}{dx} & \cdots\cdots & \int_0^L \frac{d\phi_N}{dx}\frac{d\phi_N}{dx} \end{bmatrix}_{N\times N}$$

$$\vec{C} = [C_1 \quad C_2 \quad C_3 \quad \cdots \quad C_N]^T,$$

$$\vec{F} = \begin{bmatrix} \int_0^L q(x)\,\phi_1(x)\,dx + R\phi_1(L) \\ \int_0^L q(x)\,\phi_2(x)\,dx + R\phi_2(L) \\ \vdots \\ \int_0^L q(x)\,\phi_N(x)\,dx + R\phi_N(L) \end{bmatrix}_{N\times 1}$$

## CHOICE OF BASIS FUNCTIONS:

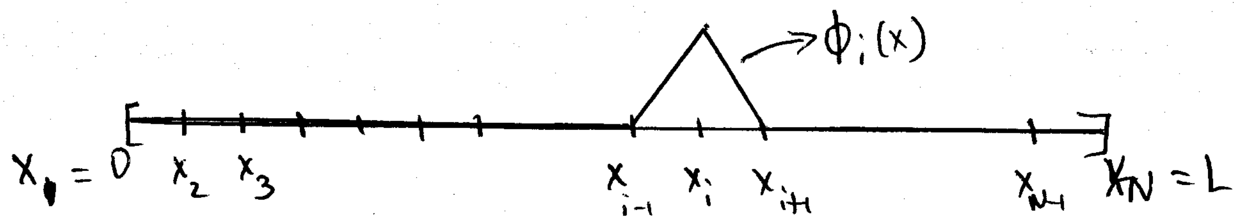
The finite element method employs basis functions that satisfy the following requirements

1. The functions $\phi_j(x)$ can be found in a systematic manner.
2. The functions $\phi_j(x)$ can be chosen such that they can be used for arbitrary domains.
3. The functions $\phi_j(x)$ have local support i.e. they are non-zero only on a small part of domain.
4. The functions $\phi_j(x)$ are piecewise polynomials
5. The functions $\phi_j(x)$ are chosen in such a way that the parameters $C_i$ defining the approximate solution $u_h(x)$ are precisely the values of the function $u_h(x)$ at the nodal points.
6. The functions $\phi_j(x)$ are bounded and continuous.

One such choice for the basis functions is

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x_{i-1} \le x \le x_i \\[2ex] \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x_i \le x \le x_{i+1} \\[2ex] 0 & \text{otherwise} \end{cases}$$

Where the interval $[0, L]$ is partitioned as
$0 = x_1 \le x_2 \le x_3 \cdots x_{i-1} \le x_i \le x_{i+1} \le \cdots \le x_{N-1} \le x_N = L$.

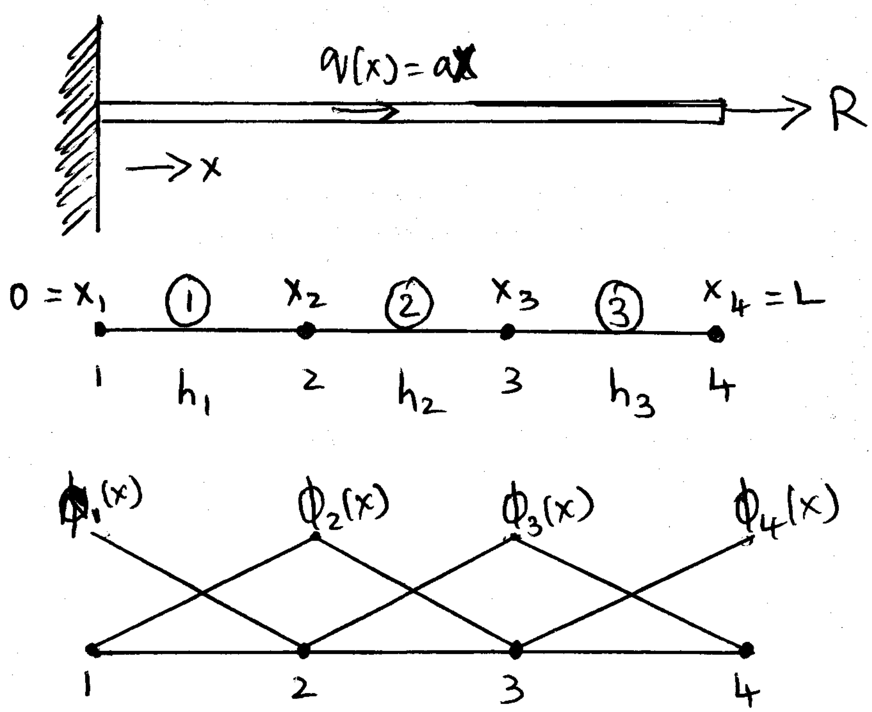



Let $h_i = x_i - x_{i-1}$ and $h_{i+1} = x_{i+1} - x_i$

we then have,

$$\frac{d\phi_i}{dx} = \begin{cases} \dfrac{1}{h_i} & x_{i-1} < x < x_i \\[2ex] \dfrac{-1}{h_{i+1}} & x_i < x < x_{i+1} \\[2ex] 0 & x < x_{i-1} \text{ and } x > x_{i+1} \end{cases}$$

Let us consider the case when the bar is partitioned into 3 elements and 4 nodes.




For this set up. the Stiffness matrix K is given by

$$K = AE \begin{bmatrix} \int_0^L \frac{d\phi_1}{dx}\frac{d\phi_1}{dx} & \int_0^L \frac{d\phi_2}{dx}\frac{d\phi_1}{dx} & \int_0^L \frac{d\phi_3}{dx}\frac{d\phi_1}{dx} & \int_0^L \frac{d\phi_4}{dx}\frac{d\phi_1}{dx} \\ \int_0^L \frac{d\phi_1}{dx}\frac{d\phi_2}{dx} & \int_0^L \frac{d\phi_2}{dx}\frac{d\phi_2}{dx} & \int_0^L \frac{d\phi_3}{dx}\frac{d\phi_2}{dx} & \int_0^L \frac{d\phi_4}{dx}\frac{d\phi_2}{dx} \\ \int_0^L \frac{d\phi_1}{dx}\frac{d\phi_3}{dx} & \int_0^L \frac{d\phi_2}{dx}\frac{d\phi_3}{dx} & \int_0^L \frac{d\phi_3}{dx}\frac{d\phi_3}{dx} & \int_0^L \frac{d\phi_4}{dx}\frac{d\phi_3}{dx} \\ \int_0^L \frac{d\phi_1}{dx}\frac{d\phi_4}{dx} & \int_0^L \frac{d\phi_2}{dx}\frac{d\phi_4}{dx} & \int_0^L \frac{d\phi_3}{dx}\frac{d\phi_4}{dx} & \int_0^L \frac{d\phi_4}{dx}\frac{d\phi_4}{dx} \end{bmatrix}$$

Let us consider a representative entry $K_{ij}$ from this matrix i.e.

$$K_{ij} = AE \int_0^L \frac{d\phi_j}{dx} \frac{d\phi_i}{dx} \, dx$$

$$= AE \left[ \int_0^{x_2} \frac{d\phi_j}{dx} \frac{d\phi_i}{dx} \, dx + \int_{x_2}^{x_3} \frac{d\phi_j}{dx} \frac{d\phi_i}{dx} \, dx + \int_{x_3}^{L} \frac{d\phi_j}{dx} \frac{d\phi_i}{dx} \, dx \right]$$

$$= AE \left[ \int_{\Omega_1} \frac{d\phi_j}{dx} \frac{d\phi_i}{dx} \, dx + \int_{\Omega_2} \frac{d\phi_j}{dx} \frac{d\phi_i}{dx} \, dx + \int_{\Omega_3} \frac{d\phi_j}{dx} \frac{d\phi_i}{dx} \, dx \right]$$

Observation :

$$K_{ij} = K_{ji} \quad \text{(Symmetric)}$$

$$K_{ij} = \sum_{k=1}^{3} AE \int_{\Omega_k} \frac{d\phi_j}{dx} \frac{d\phi_i}{dx} \, dx$$

The entries of matrix K then are:

$$K_{ii} = AE\left[\int_{\Omega_1} \frac{d\phi_i}{dx}\frac{d\phi_i}{dx}\,dx + \int_{\Omega_2} \frac{d\phi_i}{dx}\frac{d\phi_i}{dx} + \int_{\Omega_3} \frac{d\phi_i}{dx}\frac{d\phi_i}{dx}\right]$$

$$i = 1, 2, 3, 4$$

$$K_{12} = AE\left[\int_{\Omega_1} \frac{d\phi_2}{dx}\frac{d\phi_1}{dx} + \cancelto{0}{\int_{\Omega_2} \frac{d\phi_2}{dx}\frac{d\phi_1}{dx}} + \cancelto{0}{\int_{\Omega_3} \frac{d\phi_2}{dx}\frac{d\phi_1}{dx}}\right] = K_{21}$$

$$K_{13} = AE\left[\cancelto{0}{\int_{\Omega_1} \frac{d\phi_3}{dx}\frac{d\phi_1}{dx}} + \cancelto{0}{\int_{\Omega_2} \frac{d\phi_3}{dx}\frac{d\phi_1}{dx}} + \cancelto{0}{\int_{\Omega_3} \frac{d\phi_3}{dx}\frac{d\phi_1}{dx}}\right] = K_{31}$$

Similarly $K_{41} = K_{14} = K_{24} = 0$

$$K_{23} = AE\left[\cancelto{0}{\int_{\Omega_1} \frac{d\phi_3}{dx}\frac{d\phi_2}{dx}} + \int_{\Omega_2} \frac{d\phi_3}{dx}\frac{d\phi_2}{dx} + \cancelto{0}{\int_{\Omega_3} \frac{d\phi_3}{dx}\frac{d\phi_2}{dx}}\right] = K_{32}$$

$$K_{34} = AE\left[\cancelto{0}{\int_{\Omega_1} \frac{d\phi_4}{dx}\frac{d\phi_3}{dx}} + \cancelto{0}{\int_{\Omega_2} \frac{d\phi_4}{dx}\frac{d\phi_3}{dx}} + \int_{\Omega_3} \frac{d\phi_4}{dx}\frac{d\phi_3}{dx}\right] = K_{43}$$

The diagonal entries are:

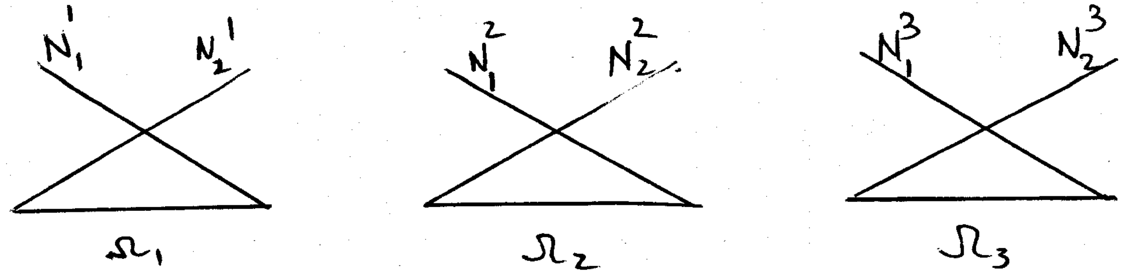
$$K_{11} = AE\left[\int_{\Omega_1} \frac{d\phi_1}{dx}\frac{d\phi_1}{dx} + \cancelto{0}{\int_{\Omega_2} \frac{d\phi_1}{dx}\frac{d\phi_1}{dx}} + \cancelto{0}{\int_{\Omega_3} \frac{d\phi_1}{dx}\frac{d\phi_1}{dx}}\right]$$

$$K_{22} = AE\left[\int_{\Omega_1} \frac{d\phi_2}{dx}\frac{d\phi_2}{dx} + \int_{\Omega_2} \frac{d\phi_2}{dx}\frac{d\phi_2}{dx} + \cancelto{0}{\int_{\Omega_3} \frac{d\phi_2}{dx}\frac{d\phi_2}{dx}}\right]$$

$$K_{33} = AE\left[\cancelto{0}{\int_{\Omega_1} \frac{d\phi_3}{dx}\frac{d\phi_3}{dx}} + \int_{\Omega_2} \frac{d\phi_3}{dx}\frac{d\phi_3}{dx} + \int_{\Omega_3} \frac{d\phi_3}{dx}\frac{d\phi_3}{dx}\right]$$

$$K_{44} = AE\left[\cancelto{0}{\int_{\Omega_1} \frac{d\phi_4}{dx}\frac{d\phi_4}{dx}} + \cancelto{0}{\int_{\Omega_2} \frac{d\phi_4}{dx}\frac{d\phi_4}{dx}} + \int_{\Omega_3} \frac{d\phi_4}{dx}\frac{d\phi_4}{dx}\right]$$

To compute these matrix entries efficiently we define local basis functions $N_k^e$ over each element $e$. For example, we have the following local basis functions



The matrix entries then become

$$K_{11} = AE \int_{\Omega_1} \frac{dN_1^1}{dx} \frac{dN_1^1}{dx} dx$$

$$K_{12} = AE \int_{\Omega_1} \frac{dN_2^1}{dx} \frac{dN_1^1}{dx} dx$$

$$K_{13} = K_{14} = 0$$

$$K_{22} = AE \left[ \int_{\Omega_1} \frac{dN_2^1}{dx} \frac{dN_2^1}{dx} dx + \int_{\Omega_2} \frac{dN_1^2}{dx} \frac{dN_1^2}{dx} dx \right]$$

$$K_{23} = AE \int_{\Omega_2} \frac{dN_2^2}{dx} \frac{dN_1^2}{dx} dx \qquad K_{24} = 0$$

$$K_{33} = AE \left[ \int_{\Omega_2} \frac{dN_2^2}{dx} \frac{dN_2^2}{dx} dx + \int_{\Omega_3} \frac{dN_1^3}{dx} \frac{dN_1^3}{dx} dx \right]$$

$$K_{34} = AE \int_{\Omega_3} \frac{dN_2^3}{dx} \frac{dN_1^3}{dx} dx$$

$$K_{44} = AE \int_{\Omega_3} \frac{dN_2^3}{dx} \frac{dN_2^3}{dx} dx$$

Define local matrices (stiffness) over each element as:

$$K^e = \begin{bmatrix} K_{11}^e & K_{12}^e \\ K_{21}^e & K_{22}^e \end{bmatrix}.$$

Where

$$K_{lk}^e = AE \int_{\Omega_e} \frac{dN_l^e}{dx} \frac{dN_k^e}{dx} \, dx$$

Then it can be noted that:

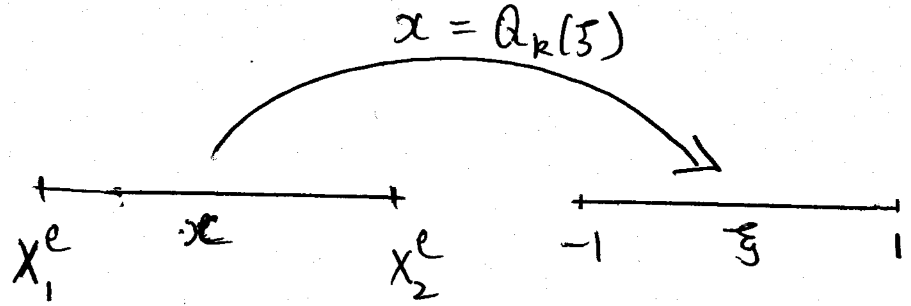
$K_{11} = K_{11}^1 \qquad K_{12} = K_{12}^1 \qquad K_{13} = 0 \qquad K_{14} = 0$

$K_{22} = K_{22}^1 + K_{11}^2 \qquad K_{23} = K_{12}^2 \qquad K_{24} = 0$

$K_{33} = K_{22}^2 + K_{11}^3 \qquad K_{34} = K_{12}^3$

$K_{44} = K_{22}^3$

Consider a linear mapping from any generic element "e" to a reference element $[-1, 1]$.



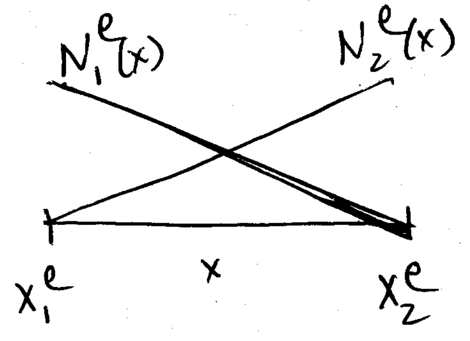

$$x = Q_k(\xi) = \frac{1-\xi}{2} X_1^e + \frac{1+\xi}{2} X_2^e$$

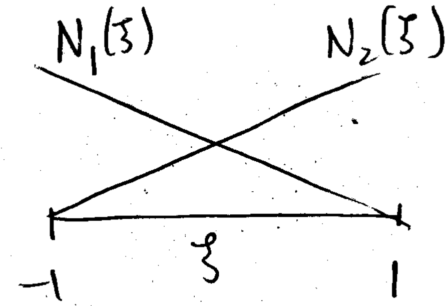
This also provides an inverse mapping

$$\xi = Q_k^{-1}(x) = \frac{2x - x_1^e - x_2^e}{x_2^e - x_1^e}$$

To determine $K^e_{lk}$, we will then use the above mapping to transform the integral from the domain $\Omega_e$ to $[-1, 1]$.

Let us now define the basis functions on $[-1,1]$ corresponding to the $N_1^e$ and $N_2^e$ on $\Omega_e$.




$$N_1^e(x) = \frac{x_2^e - x}{x_2^e - x_1^e} \qquad N_1(\xi) = \frac{1-\xi}{2}$$

$$N_2^e(x) = \frac{x - x_1^e}{x_2^e - x_1^e} \qquad N_2(\xi) = \frac{1+\xi}{2}$$

One can verify that for $x = \frac{1-\xi}{2} x_1^e + \frac{1+\xi}{2} x_2^e$ we have

$$N_1^e(x) = N_1(\xi)$$

$$N_2^e(x) = N_2(\xi)$$

Let $h_e = x_2^e - x_1^e$.

Then we have,

$$K_{11}^{e} = AE \int_{\Omega_e} \frac{dN_1^e(x)}{dx} \frac{dN_1^e(x)}{dx} \, dx$$

$$= AE \int_{-1}^{1} \left( \frac{dN_1(\xi)}{d\xi} \frac{d\xi}{dx} \right) \left( \frac{dN_2(\xi)}{d\xi} \frac{d\xi}{dx} \right) \left( \frac{x_2^e - x_1^e}{2} \right) d\xi$$

Noting that $\frac{d\xi}{dx} = \frac{2}{x_2^e - x_1^e}$ we gets

$$K_{11}^{e} = \frac{2AE}{x_2^e - x_1^e} \int_{-1}^{1} \frac{dN_1(\xi)}{d\xi} \frac{dN_1(\xi)}{d\xi} \, d\xi$$

$$= \frac{2AE}{x_2^e - x_1^e} \int_{-1}^{1} \left( \frac{-1}{2} \right) \left( \frac{-1}{2} \right) d\xi$$

$$= \frac{AE}{x_2^e - x_1^e} = \frac{AE}{h_e}$$

$$K_{12}^{e} = AE \int_{\Omega_e} \frac{dN_1^e(x)}{dx} \frac{dN_2^e(x)}{dx} dx$$

$$= AE \int_{-1}^{1} \left( \frac{dN_1(\xi)}{d\xi} \frac{d\xi}{dx} \right) \left( \frac{dN_2(\xi)}{d\xi} \frac{d\xi}{dx} \right) \left( \frac{x_2^e - x_1^e}{2} \right) d\xi$$

$$= \frac{2AE}{x_2^e - x_1^e} \int_{-1}^{1} \frac{dN_1(\xi)}{d\xi} \frac{dN_2(\xi)}{d\xi} d\xi$$

$$= \frac{AE}{x_2^e - x_1^e} \int_{-1}^{1} \left( \frac{-1}{2} \right) \left( \frac{1}{2} \right) d\xi$$

$$\Rightarrow K_{12}^{e} = \frac{-AE}{h_e} = K_{21}^{e}$$

Similarly one can verify that,

$$K_{22}^{e} = \frac{AE}{h_e}$$

We then have the element stiffness matrix

$$K^e = \frac{AE}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

To assemble these local stiffness matrices into the global matrix one must employ a element connectivity matrix that points the connection between the local basis functions over each element "e" to the global basis functions.

Global Basis Function # $\longrightarrow$

$$P = \begin{array}{c|cc} \text{element \#}\ e \downarrow & N_1^e & N_2^e \\ \hline 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 3 & 4 \end{array}$$

Using the connectivity matrix one can then assemble the individual elemental matrices into the global matrix to yield: (for uniform element size $h_e = h$):

$$\frac{AE}{h}\begin{bmatrix} 1 & -1 & & \\ -1 & 1+1 & -1 & \\ & -1 & 1+1 & -1 \\ & & -1 & 1 \end{bmatrix}$$

$$\Rightarrow K = \frac{AE}{h}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

## Computation of the right hand side (RHS)

Recall the RHS vector for the example with 3 elements and 4 nodes becomes:

$$t_j = \int_0^L q(x)\,\phi_j(x)\,dx + R\,\phi_j(L) \qquad j = 1..4$$

$$= \sum_{k=1}^{3} \int_{\Omega_k} q(x)\,\phi_j(x)\,dx + R\,\phi_j(L) \qquad j = 1..4$$

We have,

$$t_1 = \int_{\Omega_1} q(x)\,\phi_1(x)\,dx + \cancelto{0}{\int_{\Omega_2} q(x)\,\phi_1(x)} + \cancelto{0}{\int_{\Omega_3} q(x)\,\phi_1(x)} + \cancelto{0}{R\,\phi_1(L)}$$

$$t_2 = \int_{\Omega_1} q(x)\,\phi_2(x)\,dx + \int_{\Omega_2} q(x)\,\phi_2(x) + \cancelto{0}{\int_{\Omega_3} q(x)\,\phi_2(x)} + \cancelto{0}{R\,\phi_2(L)}$$

$$t_3 = \cancelto{0}{\int_{\Omega_1} q(x)\,\phi_3(x)\,dx} + \int_{\Omega_2} q(x)\,\phi_3(x) + \int_{\Omega_3} q(x)\,\phi_3(x) + \cancelto{0}{R\,\phi_3(L)}$$

$$t_4 = \cancelto{0}{\int_{\Omega_1} q(x)\,\phi_4(x)\,dx} + \cancelto{0}{\int_{\Omega_2} q(x)\,\phi_4(x)} + \int_{\Omega_3} q(x)\,\phi_4(x) + R\,\phi_4(L)$$

Using the same elementwise notations as before we have,

$$f_1 = \int_{\Omega_1} q(x)\, N_1^1(x)\, dx$$

$$f_2 = \int_{\Omega_1} q(x)\, N_2^1(x)\, dx + \int_{\Omega_2} q(x)\, N_1^2(x)\, dx$$

$$f_3 = \int_{\Omega_2} q(x)\, N_2^2(x) + \int_{\Omega_3} q(x)\, N_1^3(x)\, dx$$

$$f_4 = \int_{\Omega_3} q(x)\, N_2^3(x)\, dx + N_2^3(L)\, R$$

As before we can define local vectors as

$$f^e = \begin{bmatrix} f_1^e \\ f_2^e \end{bmatrix}$$

then

$$f_1 = f_1^1$$
$$f_2 = f_2^1 + f_1^2$$
$$f_3 = f_2^2 + f_1^3$$
$$f_4 = f_2^3 + R$$

We can then determine $f_1^e$ and $f_2^e$ as follows:

$$f_1^e = \int_{\Omega_e} q(x)\, N_1^e(x)\, dx$$

$$= \int_{-1}^{1} q\big(Q_k(\xi)\big)\, N_1(\xi) \left(\frac{x_2^e - x_1^e}{2}\right) d\xi$$

$$= \frac{h_e}{2} \int_{-1}^{1} q\big(Q_k(\xi)\big)\, N_1(\xi)\, d\xi$$

$$= \frac{h_e}{2} \int_{-1}^{1} \tilde{q}(\xi) \left(\frac{1-\xi}{2}\right) d\xi$$

Similarly $f_2^e = \int_{\Omega_e} q(x)\, N_2^e(x)\, dx$

$$= \frac{h_e}{2} \int_{-1}^{1} \tilde{q}(\xi) \left(\frac{1+\xi}{2}\right) d\xi$$

we then have the element load vector as

$$f^{e} = \frac{h_e}{2}\begin{bmatrix} \int_{-1}^{1} \tilde{q}(\xi)\left(\frac{1-\xi}{2}\right) d\xi \\ \int_{-1}^{1} \tilde{q}(\xi)\left(\frac{1+\xi}{2}\right) d\xi \end{bmatrix}$$

Assembling the load (local) vectors into the global load vector using the connectivity matrix then yields (for uniform element size $h_e = h$)

$$\frac{h}{2}\begin{bmatrix} \int_{-1}^{1} \tilde{q}(\xi)\left(\frac{1-\xi}{2}\right) d\xi \\ \int_{-1}^{1} \tilde{q}(\xi)\left(\frac{1+\xi}{2}\right) d\xi + \int_{-1}^{1} \tilde{q}(\xi)\left(\frac{1-\xi}{2}\right) d\xi \\ \int_{-1}^{1} \tilde{q}(\xi)\left(\frac{1+\xi}{2}\right) d\xi + \int_{-1}^{1} \tilde{q}(\xi)\left(\frac{1-\xi}{2}\right) d\xi \\ \int_{-1}^{1} \tilde{q}(\xi)\left(\frac{1+\xi}{2}\right) d\xi + R \end{bmatrix}$$

It must be noted that we considered an example wherein E and A were constants. However, if these were functions of x, for example then one has to retain these functions within the integral contributions. For instance,

$$K_{12}^{e} = \int_{\Omega_e} A(x)\, E(x)\, \frac{dN_1^e(x)}{dx} \frac{dN_2^e(x)}{dx}\, dx$$

$$= \frac{2}{h_e} \int_{-1}^{1} a(\xi)\, \frac{dN_1(\xi)}{d\xi} \frac{dN_1(\xi)}{d\xi}\, d\xi$$

The computations for the elemental load vector and the element stiffness matrix can be made more efficient by using Gaussian-Quadrature (2-point, 3-point etc.)

Using this, all integrals can be computed (for example using 2-point Gaussian quadrature) as

$$\int_{-1}^{1} \tilde{f}(\xi)\, d\xi = \tilde{f}\left(-\frac{\sqrt{3}}{3}\right) + \tilde{f}\left(\frac{\sqrt{3}}{3}\right)$$

## CALCULATING GLOBAL STIFFNESS MATRIX & LOAD VECTOR

Let M be the total number of elements that the domain is partitioned into and let e be a representative element. Then given the connectivity (POINTER) matrix P which is of size M x 2 (corresponding to number of elements as the row index and number of basis functions per element as column index) we have the following pseudo-algorithm

Initialize Global $K_{N \times N}$ and $f_{N \times 1}$.

For $e = 1..M$

  For $i = 1..2$

    For $j = 1..2$

      Calculate $K^e_{ij}$, $f^e_i$

      Assign $l = P(e,i)$ & $m = P(e,j)$

      $K_{lm} = K_{lm} + K^e_{ij}$

      $f_l = f_l + f^e_i$

    end

  end

end

We then obtain the global matrix system

$$K\vec{c} = \vec{F}$$

Note that prior to solving this system one must account for the essential boundary condition $u(0) = 0$. This can be done by eliminating the contribution of the basis function $\phi_1(x)$ in the above matrix system. This amounts to dropping the first row & first column of the stiffness matrix $K$ and the first row of the load vector $\vec{F}$. This condensation process then leads to the following (reduced) matrix system

$$\tilde{K}\tilde{\vec{c}} = \tilde{\vec{F}}$$

where:

$$\tilde{K} = \frac{AE}{h}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad \tilde{\vec{F}} = \begin{bmatrix} \int_{-1}^{1} \tilde{q}(\xi)\left(\frac{1+\xi}{2}\right)d\xi + \int_{-1}^{1}\tilde{q}(\xi)\left(\frac{1-\xi}{2}\right)d\xi \\ \int_{-1}^{1} \tilde{q}(\xi)\left(\frac{1+\xi}{2}\right)d\xi + \int_{-1}^{1}\tilde{q}(\xi)\left(\frac{1-\xi}{2}\right)d\xi \\ \int_{-1}^{1} \tilde{q}(\xi)\left(\frac{1+\xi}{2}\right)d\xi + R \end{bmatrix}$$

Note that from the methodology one can examine the displacement over an element "e" as $u^e = u_1^e N_1^e(x) + u_2^e N_2^e(x)$

Then one can compute the strain in the element as:

$$\varepsilon^e = \frac{du^e}{dx} = u_1^e \frac{dN_1^e}{dx}(x) + u_2^e \frac{dN_2^e}{dx}(x)$$

$$= u_1^e \left( \frac{dN_1(\xi)}{d\xi} \frac{d\xi}{dx} \right) + u_2^e \left( \frac{dN_2(\xi)}{d\xi} \frac{d\xi}{dx} \right)$$

$$= u_1^e \left( -\frac{1}{h} \right) + u_2^e \left( \frac{1}{h} \right)$$

$$= \begin{bmatrix} -\frac{1}{h} & \frac{1}{h} \end{bmatrix} \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix}$$

Then stress in element "e": $\sigma^e = E\,\varepsilon^e$