Homework 3

1. The tasks in this problem will require you to enhance the one-dimensional finite element code (*fem.m*) that was developed in the class.

   (a) Write a subroutine to evaluate the approximate value of the finite element solution at any given point $x = x_0$ that is not a node.

   (b) Write a subroutine to evaluate the approximate value of the finite element solution at any given point $x = x_0$ that is not a node.

   (c) Use the subroutine in part (a) with composite trapezoid rule that uses 300 points to estimate the error in the solution

   $$||u - u_s||_{L^2(a,b)}$$

   (d) Use the subroutine in part (b) with composite trapezoid rule that uses 300 points to estimate the error in derivative of the solution

   $$||u' - u'_s||_{L^2(a,b)}$$

   (e) Perform the following convergence study. Run the *fem.m* code with the number of intervals $MD = 2^i, i = 1, 2, 3, 4, 5, 6$. Evaluate the corresponding L^2 norm error e_i in the solution and e'_i in the derivative of the solution for each i using parts (c) and (d) respectively. Use the values of error calculated to find the rate of convegence in the L^2 norm of the solution and the derivative of the solution. Verify if the expected result matches with the theoretical results.

2. Consider the problem,

   $$\begin{aligned} u_t &= u_{xx} \quad 0 < x < 1, \quad 0 < t < \infty \\ u(0,t) &= 0 \qquad 0 < t < \infty \\ u(1,t) &= 0 \qquad 0 < t < \infty \\ u(x,0) &= \sin(\pi x) \quad 0 \le x \le 1 \end{aligned}$$

   Adapt the MATLAB code (*fem.m*) to accomodate time-dependence and solve the given problem from times $t = 0$ to $t = 1$. Fix the number of intervals MD $= 8$ and use the stepsize in time $\Delta t = (0.1)^n$ for $n = 1, 2, 3, 4$. For each n, plot the exact solution calculated at 300 points in the interval $[0, 1]$ and compare it against the finite element solution calculated over the 8 intervals for the final time. Also, evaluate the corresponding L^2 norm error e_n for $n = 1, 2, 3, 4$ between the exact and finite element solutions at the final time. Explain your observations.