Homework 2

1. Consider the (strong form) problem $-u''(x) = \delta(1)$ in $(0,1), u(0) = 0, u'(1) = 0$ with a point load at $x = 1$, i.e. $f = \delta(1)$, the delta function. Define $M(v) = \frac{1}{2}(v', v') - v(1)$ and the space $V = \left\{ v \in L^2(0,1) : \int_0^1 (v')^2 \; dx < \infty, v(0) = 0 \right\}$. Consider the minimization problem: Find $u \in V$ such that $M(u) \leq M(v)$ for all $v \in V$.
   (a) Show that $u(x)$ is a solution to the strong form if and only if it is also a solution to the minimization problem.
   (b) Show that $u(x) = x$ is a solution to the minimization problem. Describe your observation about this solution corresponding to the function f being singular at $x = 1$.

2. Let $x \in I = (0,1)$.
   (a) Let $v(x) = x^{-\alpha}$. Find the admissible values of α for which $v \in L^2(I)$.
   (b) Let $v(x) = (1-x)^{2.51}$. Find the largest integer value of k for which $v \in H^k(I)$. Will $v(x) \in C^{(k)}(I)$ for this value of k? (Show all calculations).

3. Consider the two-point BVP: $-u''(x) = f(x)$ in $(0,1)$ with $u(0) = 0, u(1) = 0$. Let
$$V = \left\{ v : v \text{ is defined on } I = (0,1), \int_0^1 (v)^2 + (v')^2 \; dx < \infty, v(0) = v(1) = 0 \right\}$$
and $f \in L^2(I)$ where $I = (0,1)$ with the scalar product: $(v, w)_V = \int_0^1 (v\; w + v'\; w') \; dx$ and the corresponding norm: $||v||_V = \left(\int_0^1 v^2 + (v')^2 \, dx \right)^{\frac{1}{2}}$. The strong form can be reformulated to yield the weak form: Find $u \in V$ such that $a(u,v) = L(v)$ for all $v \in V$ where $a(u,v) = \int_0^1 u'(x)\; v'(x) \; dx$ and $L(v) = \int_0^1 f(x)\; v(x) \; dx$.
   (a) Show that $||.||_V$ is a norm by definition.
   (b) Verify that the bilinear functional $a(u,v)$ is symmetric, continuous and elliptic.

4. Consider the weak-form: Find $u \in V$ such that: $a(u,v) = (f,v)$ for all $v \in V$ where,
$$\begin{aligned} a(u,v) &= \int_0^1 (u''v'' + u'v') \; dx \qquad\qquad (f,v) = \int_0^1 f\; v \; dx + \sin(1)\; v'(1) \\ V &= \left\{ v : v \text{ is defined on } I = (0,1), \int_0^1 (v'')^2 + (v')^2 \; dx < \infty, \; v(0) = v(1) = 0 \right\} \end{aligned}$$
Derive the associated (strong form) boundary value problem.

5. Consider the two-point boundary value problem:
$$-((1+x)u')' = 0 \text{ in } (0,1), u(0) = 0, u'(1) = 0$$
Partition the interval $0 < x < 1$ into 3 subintervals of equal length and let $\mathtt{S}$ be a corresponding space of continuous piecewise linear functions vanishing at $x = 0$.
   (a) Use the space $\mathtt{S}$ to formulate the finite element method.
   (b) Find the stiffness matrix $\mathbf{K}$ and the load vector $\mathbf{b}$.
   (c) Show that stiffness matrix $\mathbf{K}$ is symmetric, tridiagonal and positive definite.