Analysis of the Finite Element Method
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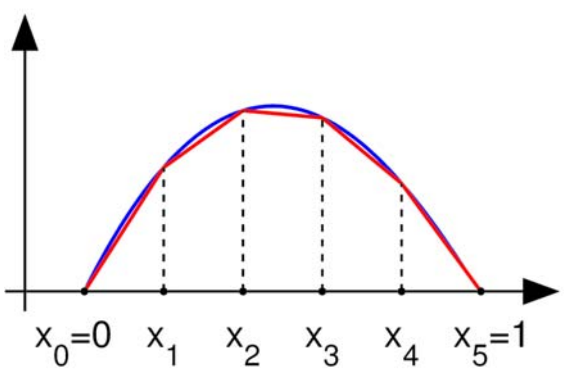

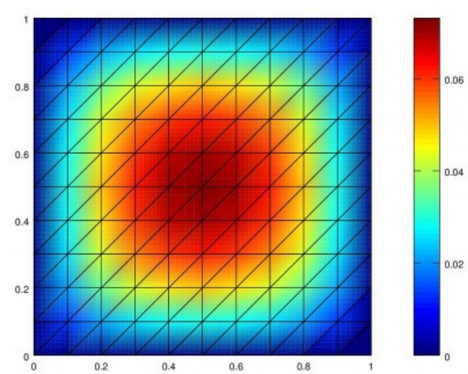

$$A = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}$$

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Strong $\Leftrightarrow$ Weak $\Leftrightarrow$ Minimization

- **Strong Form**: Let $f \in C^0([0,1])$. Find $u \in C^2([0,1])$

$$\boxed{\begin{array}{c} -u''(x) = f(x) \quad \text{in} \ (0,1) \\ u(0)=0 \qquad u'(1)=0 \end{array}}$$

- **Weak Form**: Find $u \in V$ such that: $a(u,v) = (f,v)$ for all $v \in V = \{ v \in L^2(0,1) : a(v,v) < \infty \text{ and } v(0)=0 \}$ we have,

$$\boxed{a(u,v) = \int_0^1 u'(x)v'(x)\,dx \quad \text{and} \quad (f,v) = \int_0^1 f(x)v(x)\,dx}$$

- **Minimization Problem**: Find $u \in V$ such that:

$$\boxed{M(u) \le M(v) \ \text{ for all } v \in V}$$

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Ritz-Galerkin Approximation

- Let $S \subset V$ be any finite dimensional subspace.

$$S = \text{span}\{\phi_1(x), \phi_2(x), \ldots\ldots, \phi_N(x)\}$$

- Weak form on finite dimensional space:
  – $\boxed{\text{Find } u_S \in S \text{ such that: } a(u_S, v) = (f, v) \quad \text{for all } v \in V}$

- Choosing $v = \phi_i(x)$ for any $i = 1..n$ and $\boxed{u_S(x) = \sum_{j=1}^{N} c_j \phi_j(x)}$ we have

  - Find $\vec{c} = \{c_j\}$ that satisfies the matrix system: $\boxed{K\,\vec{c} = \vec{f}}$

  where,

$$\boxed{\begin{array}{ll} K_{ij} = a(\phi_j, \phi_i) & i, j = 1..n \\ f_i = (f, \phi_i) & i = 1..n \end{array}}$$

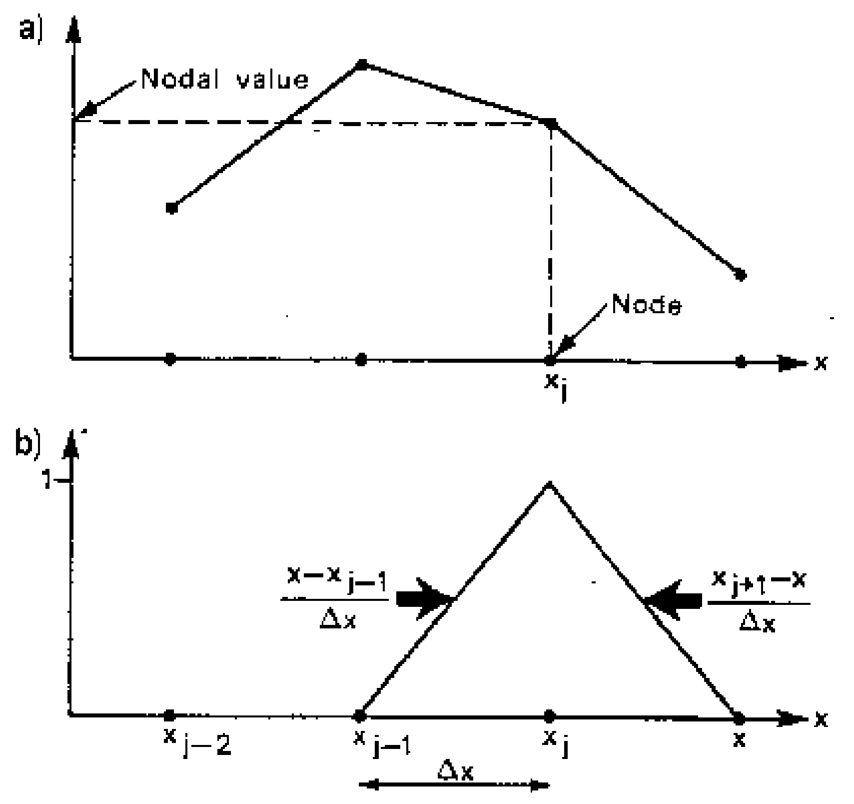
Linear Basis Functions

- Partition [0, 1]

$$0 = x_0 < x_1 < ..... < x_{j-1} < x_j < x_{j+1} < .... < x_M = 1$$

$$h_j = x_j - x_{j-1} \text{ (Non - uniform grid)}$$

$$\Delta x = h_j \text{ (Uniform grid)}$$



- For j=1,..n, let

$$\phi_j(x) = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

$$u_S(x_i) = \sum_{j=1}^{N} c_j \phi_j(x_i) = c_i$$

Estimates for $\| u - u_s \|_V$

- Theorem: If u'' is continuous then,

$$\| u - u_s \|_V \le c\, h \max | u''(x) |$$

- Key steps in proof
  - Consider the piecewise linear interpolant u_I and prove

$$\max | u(x) - u_I(x) | \le \frac{h^2}{8} \max | u''(x) |$$

$$\max | u'(x) - u_I'(x) | \le h \max | u''(x) |$$

  - Estimate $\| u - u_s \|_V$ using the above results to prove theorem.

Need a better result

- The factor "h" in the estimate reflects the observed rate of convergence as mesh is refined.
- The imperfection is $\max |u''(x)|$. It is not satisfactory to have to assume that u'' is continuous or even that it is bounded when we require a result in the energy norm.
- So we look for an estimate that only assumes that u'' has finite energy in the L^2 norm.

Estimates for $\| u - u_s \|_{H^1(0,1)}$

- Theorem: If $u'' \in L^2$ then,

$$\| u - u_s \|_{H^1(0,1)} \leq \frac{h}{\pi}\left(1 + \frac{h^2}{\pi^2}\right)^{\frac{1}{2}} \| u''(x) \|_{L^2(0,1)}$$

- Key steps in proof
  - Consider the piecewise linear interpolant u_I and prove

$$\| u - u_I \|_{L^2(0,1)} \leq \frac{h^2}{\pi^2} \| u'' \|_{L^2(0,1)}$$

$$\| u' - u_I' \|_{L^2(0,1)} \leq \frac{h}{\pi} \| u'' \|_{L^2(0,1)}$$

  - Estimate $\| u - u_s \|_{H^1(0,1)}$ using the above results to prove theorem.