Analysis of the Finite Element Method
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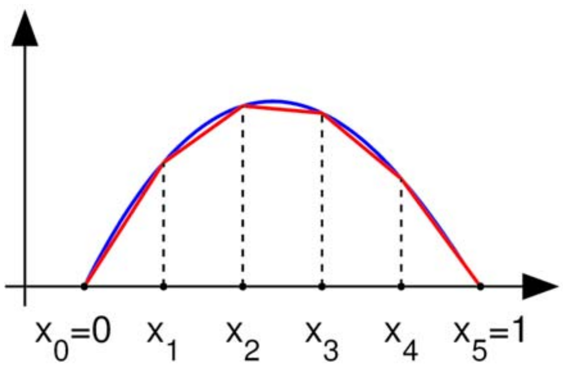

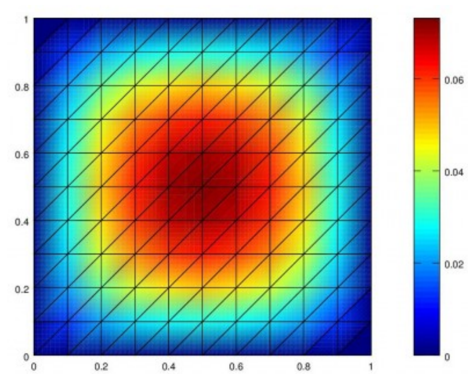

$$A = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}$$

Examples of PDEs

- Heat Equation
- Wave Equation
- Laplace Equation
- Advection Equation
- Convection-Diffusion-Reaction Equation
- Maxwells equation
- Schrödinger equation
- Black–Scholes equation

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Mathematical Formulation

- Consider the two-point BVP

$$-u''(x) = f(x) \quad \text{in} \;\; (0,1)$$

$$u(0) = 0$$

$$u'(1) = 0$$

- Define:
  - Finite Energy $\int_0^1 \left(f(x)\right)^2 dx < \infty$
  - Scalar Product $(v, w) = \int_0^1 v(x)\, w(x)\, dx$

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Deriving the “weak” variational form

- Let $v(x)$ be any (sufficiently regular) function such that $v(0) = 0$.
- Multiplying the BVP by the test function $v(x)$ and integrating by parts yields,

$$a(u,v) = (f,v)$$

where,

$$a(u,v) = \int_0^1 u'(x)\, v'(x)\, dx$$

$$(f,v) = \int_0^1 f(x)\, v(x)\, dx$$

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Strong Form $\Longrightarrow$ Weak Form

- Define the space:

$$V = \left\{ v \in L^2(0,1) : a(v,v) < \infty \text{ and } v(0) = 0 \right\}$$

- Our “weak” problem

  – Find $u \in V$ such that: $a(u,v) = (f,v)$ for all $v \in V$

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Weak Form → Strong Form

- **Strong** Form:
  - Suppose $f \in C^0([0,1])$ then find $u \in C^2([0,1])$

$$-u''(x) = f(x) \quad \text{in} \quad (0,1)$$
$$u(0) = 0$$
$$u'(1) = 0$$

- **Weak** Form:
  - Find $u \in V$ such that: $a(u,v) = (f,v)$ for all $v \in V$

- **Theorem**: Suppose $f \in C^0([0,1])$ and $u \in C^2([0,1])$ satisfies the weak form, then u solves the strong form.

Boundary Conditions

- DIRICHLET
  - **Essential** Boundary Condition
  - Appears **explicitly**

$$u(0) = 0$$

- NEUMANN
  - **Natural** Boundary Condition
  - Incorporate **implicitly**

$$u'(1) = 0$$

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Ritz-Galerkin Approximation

- Let $S \subset V$ be any finite dimensional subspace.

- Weak form on finite dimensional space:
  - Find $u_S \in S$ such that: $a(u_S, v) = (f, v)$ for all $v \in V$

- **Theorem**: Given $f \in L^2(0,1)$ then the above weak form on the finite dimensional subspace S has a **unique** solution.

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Matrix Formulation

- Let $S = \mathrm{span}\{\phi_1(x), \phi_2(x), \ldots\ldots, \phi_N(x)\}$ and $u_S(x) = \sum_{j=1}^{N} c_j \phi_j(x)$
- Choosing $v = \phi_i(x)$ for any $i = 1..n$ the weak form

  - Find $u_S \in S$ such that: $a(u_S, v) = (f, v)$ for all $v \in V$

  becomes:

  - Find $\vec{c} = \{c_j\}$ that satisfies the matrix system:

$$K\vec{c} = \vec{f}$$

where,

$$K_{ij} = a(\phi_j, \phi_i) \qquad i, j = 1..n$$

$$f_i = (f, \phi_i) \qquad i = 1..n$$

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Properties of Stiffness Matrix

- K is symmetric

$$a(\phi_i, \phi_j) = a(\phi_j, \phi_i)$$

- K is positive definite

$$\vec{v}^{\,T} K \vec{v} > 0 \quad \text{for } \vec{v} \neq 0$$

Weak Form $\Longleftrightarrow$ Minimization Problem

- Define the linear functional $M : V \rightarrow \Re$

$$M(v) := \frac{1}{2}(v', v') - (f, v)$$

- **Minimization** Problem: Find $u \in V$ such that:

$$M(u) \leq M(v) \ \text{ for all } v \in V$$

- Recall the **Weak** formulation:
  - Find $u \in V$ such that: $a(u, v) = (f, v)$ for all $v \in V$

- **Theorem**: The minimization problem and weak formulation have the **same** solution.

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Strong ⟺ Weak ⟺ Minimization

- **<u>Strong</u>** <u>Form</u>: Let $f \in C^0([0,1])$. Find $u \in C^2([0,1])$

$$-u''(x) = f(x) \quad \text{in} \ \ (0,1)$$
$$u(0) = 0$$
$$u'(1) = 0$$

- **<u>Weak</u>** <u>Form</u>: Find $u \in V$ such that: $a(u,v) = (f,v)$ for all $v \in V$ where,

$$a(u,v) = \int_0^1 u'(x)v'(x)\,dx \quad \text{and} \quad (f,v) = \int_0^1 f(x)v(x)\,dx$$

- **<u>Minimization</u>** <u>Problem</u>: Find $u \in V$ such that:

$$M(u) \le M(v) \ \ \text{for all}\, v \in V$$