$$u_t + u u_x = u_{xx} \qquad 0<x<1\,,\ t \geq 0$$
$$u(0,t) = u(1,t) = 0 \qquad t \geq 0 \qquad (1)$$
$$u(x,0) = \varphi(x) \qquad 0<x<1$$

$$u_t + \frac{1}{2}(u^2)_x = u_{xx} \qquad 0<x<1\,,\ t \geq 0$$
$$u(0,t) = u(1,t) = 0 \qquad t \geq 0$$
$$u(x,0) = \varphi(x) \qquad 0<x<1$$

Let $v : [0,1] \longrightarrow \mathbb{R}$
$x \longmapsto v(x)$

$$u_t v + \frac{1}{2}(u^2)_x v = u_{xx} v$$
$$u(0,t) = u(1,t) = 0$$
$$u(x,0) = \varphi(x)$$

$$\int_0^1 u_t v\,dx + \frac{1}{2}\int_0^1 (u^2)_x v\,dx = \int_0^1 u_{xx} v\,dx$$
$$u(0,t) = u(1,t) = 0$$
$$u(x,0) = \varphi(x)$$

Integration by parts:

$$\int_0^1 u_t v\,dx + \frac{1}{2}\int_0^1 (u^2)_x v\,dx = -\int_0^1 u_x v_x\,dx + u_x v\Big|_0^1$$

lets assume that

$v = 0$ at $x=0$ and $x=1$ then

$$\int_0^1 u_t v \, dx + \frac{1}{2}\int_0^1 (u^2)_x v \, dx = -\int_0^1 u_x v_x \, dx \qquad (2)$$

So, if $u(x,t)$ satisfies (1) then $u(x,t)$ satisfies (2) for every $v(x)$ with $v(0)=v(1)=0$

Conversely, if $u(x,t)$ with $u(0,t)=u(1,t)=0$ satisfies (2) for every $v(x)$ with $v(0)=v(1)=0$ then

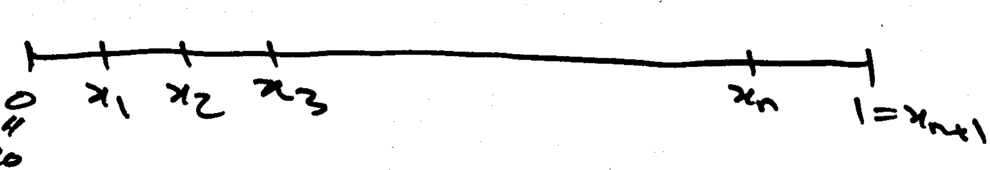
$$u_t + u u_x = u_{xx} \qquad 0<x<1, \quad t \geq 0$$
$$u(0,t) = u(1,t) = 0 \qquad t \geq 0$$

for each fixed t.

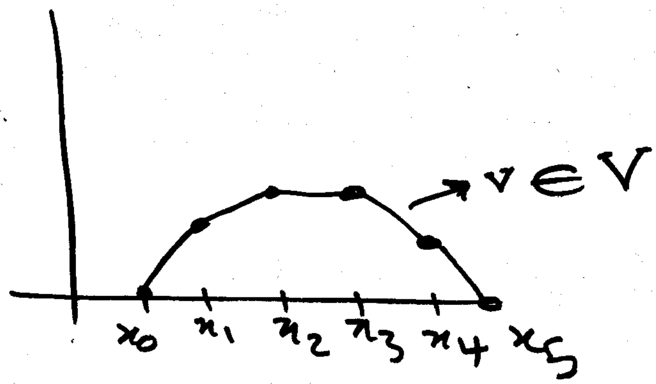
Let $H := \{ v : [0,1] \longrightarrow \mathbb{R} \}$
continuous and "a little more" and $v(0)=v(1)=0$

(fix t) $\Rightarrow$ we are searching for $u(x,t) \in H$ such that (2) is satisfied for every $v \in H$.

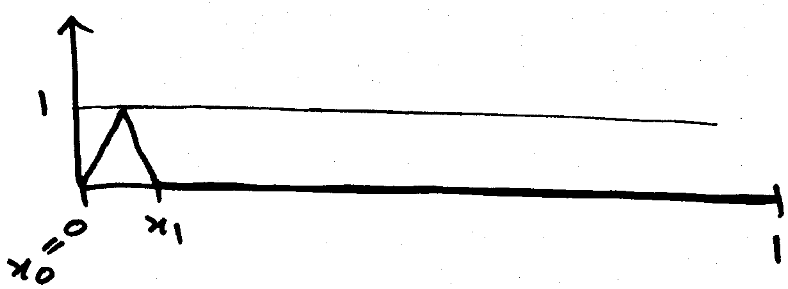
But H is infinite dimensional. $\text{span}\{\sin(n\pi x)\}_{n=1\ldots\infty}$ linearly independent. (Not good for numerical purposes.) So we wish to restrict our search space $\underbrace{}_{\text{for solution}}$ to a finite dimensional one $V \subseteq H$.



$$V := \{ v : [0,1] \longrightarrow \mathbb{R} : v \text{ is continuous and } v\big|_{[x_k, x_{k+1}]} \text{ is linear for } k = 0, \ldots, n \text{ and } v(0) = v(1) = 0 \}$$



V is finite dimensional since 8



form a basis for V.

$$v_k(x) = \begin{cases} \dfrac{x - x_{k-1}}{x_k - x_{k-1}} & x \in [x_{k-1}, x_k] \\[2ex] \dfrac{x_{k+1} - x}{x_{k+1} - x_k} & x \in [x_k, x_{k+1}] \\[2ex] 0 & \text{otherwise} \end{cases} \qquad k = 1, \ldots, n$$

find $u(x,t)$ ~~...~~ such that for each fixed t, $u(x,t) \in V$ and ~~...~~ for each $v \in V$, $u(x,t)$ satisfies:

$$\begin{cases} \int_0^1 u_t v \, dx + \frac{1}{2}\int_0^1 (u^2)_x v \, dx = -\int_0^1 u_x v_x \, dx \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall t \geq 0 \\ \int_0^1 u(x,0)\, v(x)\, dx = \int_0^1 \varphi(x) v(x)\, dx \end{cases}$$

$u(x,t) \in V \Rightarrow$

$$u(x,t) = \sum_{i=1}^{N} c_i(t) V_i(x)$$

$\boxed{N = \text{dimension of } V}$

$$\Rightarrow u_t = \sum_{i=1}^{N} \dot{c}_i(t) V_i(x)$$

$$[u(x_j,t)]^2 = \left[\sum_{i=1}^{N} c_i(t) V_i(x_j)\right]^2 =$$

$$c_i^2(t) V_i(x_i) = \sum_{i=1}^{N} c_i^2(t) V_i(x_j)$$

Let's assume that:

$$[u(x,t)]^2 = \sum_{i=1}^{N} c_i^2(t) V_i(x)$$

$\therefore$

$$\begin{cases} \sum_{i=1}^{N} \dot{c}_i(t) \int_0^1 V_i V_j + \frac{1}{2} \sum_{i=1}^{N} c_i^2(t) \int_0^1 V_i' V_j = -\sum_{i=1}^{N} c_i(t) \int_0^1 V_i' V_j' \\ \sum_{i=1}^{N} c_i(0) \int_0^1 V_i V_j = \int_0^1 \varphi(x) V_j(x)\, dx \qquad j = 1, \dots, N \end{cases}$$

$$M := \left[\int_0^1 v_i v_j\right]_{i,j=1}^{N} \quad \text{and} \quad K := \left[\int_0^1 v_i' v_j'\right]_{i,j=1}^{N}$$

and $$A := \left[\int_0^1 v_i v_j'\right]_{i,j=1}^{N}$$

$$M = \frac{h}{6}\begin{bmatrix} 4 & 1 & & & 0 \\ 1 & 4 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & 4 \end{bmatrix} \qquad \boxed{h = x_i - x_{i-1}}$$

$$A = \begin{bmatrix} 0 & 1/2 & & \\ -1/2 & 0 & 1/2 & \\ & -1/2 & \ddots & \ddots \\ & & \ddots & \ddots & 1/2 \\ & & & -1/2 & 0 \end{bmatrix}$$

$$K = \frac{1}{h}\begin{bmatrix} 2 & -1 & & 0 \\ -1 & 2 & -1 & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ 0 & & & -1 & 2 \end{bmatrix}$$

let $$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}$$

$$\begin{cases} M\dot{c} = -Kc - (1/2)A\,(c.\char`^2) \\ Mc(0) = U_0 = \left[\int_0^1 e(x)\, v_j(x) \right]_{j=1}^{N} \end{cases}$$

→ a system of ordinary differential equations

M is invertible (believe)

$$\begin{cases} \dot{c} = M^{-1}\left(-Kc - \tfrac{1}{2}A\,(c.\char`^2)\right) \\ c(0) = M^{-1}U_0 \end{cases}$$

use one ode solver like ode15s, ode23 ode45 of Matlab.

$$\Rightarrow \quad c(t) = \begin{bmatrix} c_1(t) \\ \vdots \\ c_N(t) \end{bmatrix} \quad \text{are computed}$$

$$\Rightarrow \boxed{u(x,t) = \sum_{i=1}^{N} c_i(t)\, v_i(x)}$$