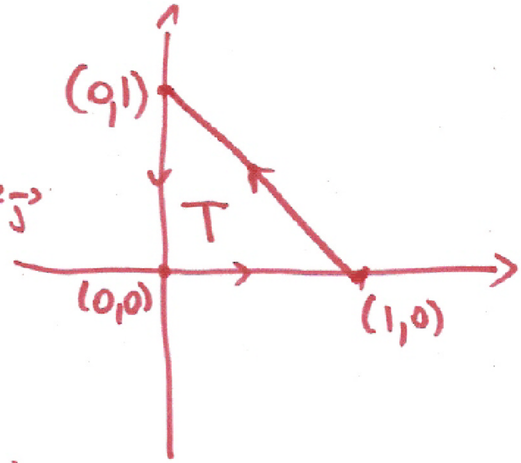
Problem 10. (10 pts) Apply Green's Theorem to evaluate

$$\oint_C \vec{F} \cdot d\vec{r}$$

where C is the triangle bounded by $x = 0, x + y = 1, y = 0$, (oriented counter-clockwise) and $\vec{F}(x, y) = y^2\vec{i} + x^2\vec{j}$.



LET T BE THE INSIDE OF THE TRIANGLE, AND WRITE $\vec{F} = P\vec{i} + Q\vec{j} = y^2\vec{i} + x^2\vec{j}$

THEN BY GREEN'S THM,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_T (Q_x - P_y)\, dA(x,y)$$

$$= \iint_T (2x - 2y)\, dA(x,y) \quad \left(= 0 \text{ CLEARLY BY SYMMETRY ABOUT } Y = X\right)$$

$$= \int_0^1 \int_0^{1-x} (2x - 2y)\, dy\, dx$$

$$= \int_0^1 (2xy - y^2)\Big|_{y=0}^{y=1-x} dx = \int_0^1 2x(1-x) - (1-x)^2 dx$$

$$= \int_0^1 2x - 2x^2 - 1 - x^2 + 2x\, dx = \int_0^1 4x - 3x^2 - 1\, dx$$

$$= 2x^2 - x^3 - x \Big|_0^1 = 2 - 1 - 1 = \boxed{0}$$