Problem 7. (10 pts) Consider the vector field $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$. Find the flux of $\vec{F}$ through the surface of the cone $\{z > 0, z = 1 - \sqrt{x^2+y^2}\}$ oriented by the normals that have a positive dot product with $\vec{k}$.

WE USE THE DIVERGENCE THM.

LET $\mathcal{S} = \{z > 0,\ z = 1 - r\}$ ORIENTED AS ABOVE

$D^+ = \{z = 0,\ x^2+y^2 \leq 1\}$ ORIENTED UP.

$E = \{z > 0,\ z \leq 1 - r\}$ THE SOLID CONE

THEN: $$\iint_{\mathcal{S}} \vec{F}\cdot d\vec{S} = \iint_{D^+} \vec{F}\cdot d\vec{S} + \iiint_E \operatorname{div}\vec{F}\, dV$$

Now $$\iint_{D^+} \vec{F}\cdot d\vec{S} = \iint_{D^+} \vec{F}\cdot\vec{k}\, dS = \iint_{D^+} z^2\, dS = 0 \quad (z=0)$$

AND $$\iiint_E \operatorname{div}\vec{F}\, dV = \iiint_E (2x+2y+2z)\, dV$$

$$= \int_0^{2\pi}\int_0^1\int_0^{1-r} (2r\cos\theta + 2r\sin\theta + 2z)\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1\int_0^{1-r} 2zr\, dz\, dr\, d\theta$$

$$= 2\pi\int_0^1 r(1-r)^2\, dr = 2\pi\int_0^1 r^3 - 2r^2 + r\, dr$$

$$= 2\pi\left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right) = \boxed{\frac{\pi}{6}}$$