RESEARCH STATEMENT

NEIL EPSTEIN

My research concerns commutative algebra, the study of commutative rings, i.e. systems that admit well-behaved operations of addition, multiplication and subtraction. Commutative algebra is a central topic in modern mathematics. Most of my work has focused on fundamental problems within commutative algebra, broadly broken into the topics of (1) properties of ring extensions and modules, and (2) closure operations on ideals and submodules. Recently I am also involved in multidisciplinary work (3) algebraic statistical methods applied to biology. I have an energetic and wide-ranging research program, with many collaborators representing several distinct research projects, some of which are outlined below. (In the sequel, I have bolded my coauthors and graduate student advisees at their initial mentions.)

Ring extensions and modules. Work of mine involving ring extensions concentrates on ring maps that are somehow tame, in that they have minimal or well-controlled effect on the ideals and modules over the respective rings. This work is represented by the articles [ES16a, ES16b, EN13, EN14]. In [ES16a], **Jay Shapiro** (George Mason University) and I generalize a 19th century theorem [Ded92, Mer92] on polynomials to the case of power series, correcting a 36-year old error [Rus78] in the literature. Our articles [ES16b, ES17, ESa] elaborate these ideas further, based on the framework of Ohm and Rush in [OR72, Rus78]. In [EN13], **Peyman Nasehpour** (University of Tehran) and I develop the tool of the *Armendariz map* to control the behavior of the *zero-divisor graph*, a graph associated to algebraic objects [AL99, DMS02], with consequences for algebra and topology. Later I advised the M.S. thesis of **Anna-Rose Wolff** relating to such graphs [Wol15]; she has since matriculated at Purdue to pursue a Ph.D. **Hop Nguyen** (University of Genoa, Italy) and I show in [EN14] that the algebraic notion of *retraction* corresponds in many cases to chopping off parts of a stick-and-ball diagram called the *Stanley-Reisner complex*.

In joint work with **Shapiro** [ES16c], we find a new class of integral domains, the *perinormal* domains, and we fit them nicely into existing commutative algebra contexts. Perinormality is characterized by the *nonexistence* of certain types of ring extensions. We show that any Noetherian normal domain (or even any Krull domain) is perinormal, and that any perinormal domain is weakly normal, seminormal, and satisfies (R_1). We give a geometric-type characterization in the context of universally catenary domains. In further work [ESb], we obtain constructions that yield perinormal domains in a variety of contexts. This work has attracted attention in the form of two papers [DR15, DR16] the first of which shows the property to hold in even more generality than we suspected, and the second of which extends the property beyond the context where we defined it.

Regarding modules, I have concentrated on the important homological properties of flatness and injectivity, characterizing both with **Yongwei Yao** (Georgia State University) in the Noetherian case [EY12] and flatness with **Shapiro** in the general case [ES14b]. Our characterizations are surprising, in that we characterize flatness (and injectivity) in purely ring-theoretic (rather than homological) terms, using *(co)associated primes* in the Noetherian case and *strong Krull primes* in the general case.

Closure operations. A *closure operation* on the ideals of a ring R may be thought of as a way to replace an arbitrary ideal by a larger ideal that may

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have better properties than the first ideal (e.g. the quotient ring of the closed ideal may be better behaved, or the closed ideal may be easier to compute than the original). Popular examples include tight closure [HH90], integral closure [NR54], and the radical operation [cf. any introduction to algebraic geometry]. There are many ways to think about closure operations; see my survey article [Eps12].

Early on, I found novel methods to break up the closure of an ideal into bite-size pieces (e.g. in my articles [Eps05, Eps10]). This work led Vraciu to her prime characteristic analogue [Vra06] of Watanabe's algebraic varieties associated to arbitrary ideals in a normal domain. My work with **Holger Brenner** (University of Osnabrück, Germany) puts some of these pieces into the context of the powerful tool of local cohomology [BE].

Some of my joint work involves *Hilbert-Kunz multiplicity* and its variants. Hilbert-Kunz multiplicity [Mon83] has historically been aimed at measuring how singular a ring is [WY00] or how far two nested finite colength ideals are from having the same tight closure [HH90]. In a long-standing project with **Yao**, we explore several different generalizations of Hilbert-Kunz multiplicity to ideals that are *not* of finite colength. Indeed, our definition of *generalized* Hilbert-Kunz multiplicity [EY17] was one of the key technical tools used in Brenner's solution in the negative [Bre13] of the longstanding question, open since Hilbert-Kunz multiplicity was shown in 1983 to exist [Mon83], of whether Hilbert-Kunz multiplicity is always rational. In some cases, we get sufficient or necessary conditions for a nested pair of ideals to share the same tight closure. One of our generalizations [EY] gives rise to a *different* operation (i.e. apart from ordinary tight closure) on ideals that we call the "unmixed tight closure" of an ideal, hitherto unknown. Another perspective arises from joint work [EV16] with **Javid Validashti** (Cleveland State), where we obtain bounds and estimates of the Hilbert-Kunz multiplicity of a product IJ of ideals in terms of the Hilbert-Kunz multiplicities of I and J separately. We also provide criteria for when a *non*-nested pair of ideals have the same tight closure.

I have also contributed to combinatorial interpretations (cf. [BE11] with **Joseph Brennan**, University of Central Florida) and homological analysis [Eps07, Eps06] of certain closure operations. In the latter, I solved a problem of Aberbach from 1994 [Abe94] of whether phantom regular sequences are all of the same length, an ingredient in the analysis of the localization question in tight closure theory.

In [ES14a], **Karl Schwede** (University of Utah) and I constructed a *dual* to tight closure that put many old results into a new light. One of the somewhat mysterious objects in tight closure theory has been the *test ideal*; we showed that with our new operation of *tight interior*, the test ideal is just the tight interior of the ring itself. We use this to recover many results about the test ideal and generalize the results to tight interiors of arbitrary modules.

My Ph.D. students, **Thomas Ales** and *George Whelan* have been doing their thesis work on closure operations as well. Whelan in particular found a link between associated primes to Frobenius closures of Frobenius powers of ideals and associated primes in a non-Noetherian model (the *perfect closure*) of the base ring. Ales has been analyzing tight closure and star-spread in commutative rings related to combinatorics.

In [Eps15], I interpreted an old idea (that of *semistar operations* [MO94]) in terms of closure operations. It turns out that many closure operations of interest (e.g. Frobenius closure, tight closure, integral closure, plus closure) are what I call *standard* and of finite type. I showed that there is a one-to-one correspondence between finite-type semistar operations on a ring and finite-type standard closure operations on its ideals.

I have also been involved in the invention of new and useful closure operations altogether [EHa, EHb, EU] that point out subtle distinctions in ways that the old closure operations could not. In [EHa], **Mel Hochster** (University of Michigan) and I analyze the geometric notion of continuous closure [Bre06] algebraically, showing among other things that certain polynomials can be represented as interesting combinations of continuous functions and other polynomials. In [EHb], Hochster and I tackle the fact [BM10] that tight closure typically has bad geometric properties by constructing a variant (*homogeneous* tight closure) whose geometric properties are better behaved. We show that in almost all cases where tight closure had been known to behave well geometrically, it coincides with homogeneous tight closure, hence providing a *reason* for the good behavior. We hope to use this new framework to attack open problems in tight closure theory, particularly the question of whether weak F-regularity localizes. We are also attacking the problem from another angle by introducing *strongly closed ideals* [EHc].

In [EU], **Bernd Ulrich** (Purdue) and I introduce the notion of the *liftable integral closure* of a submodule. This differs from the ordinary definitions of integral closure of submodules (and we compare it carefully to existing notions), but it has certain advantages. In particular, we can use it to find a large class of rings R such that for any Artinian R-module M, there are torsionless R-modules $L \subseteq T$ such that T is integral over L and $M \cong T/L$. One can think of this as "representing" an Artinian module by an integral extension of modules. **Craig Huneke** (University of Virginia) and I have extended this latest result to let T, L be ideals.

Algebraic statistics in biology. Quite recently, I have begun conversations with biologists to apply commutative algebra to genomics. In particular, **Iosif Vaisman** (GMU) and I propose to apply my expertise in commutative algebra and combinatorics to mutagenic analysis to deduce shapes of proteins and develop drugs to combat HIV virus and cancer growth. In furtherance of this, my graduate student Thomas Ales will use a Provost Fellowship grant to spend a portion of his time in the 2017-18 academic year mixing commutative algebra and bioinformatics. I have attended a Hackathon in February and an idea testing session for the 2017 Multidisciplinary Seed Funding Program in April. Vaisman and I ran a training session on the subject as part of Mason Modeling Days at the end of May.

Algebraic geometry has been used for statistical analysis of genomes for at least 12 years. See [PS05]. However, it has rarely been approached from a truly algebraic perspective such as mine, nor from an expert in ring extensions or closure operations. Thus, I expect we will develop both new math and new biology in these pursuits.

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