Rational Functions as Sums of Reciprocals of Polynomials

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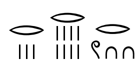
Abstract. An algorithm is given to express a proper rational function (i.e., the degree of the denominator exceeds that of the numerator) over any field as an alternating sum of reciprocals of polynomials of increasing degree. This is analogous to existing algorithms for representing proper rational numbers.

1. INTRODUCTION. Since ancient Egyptian times, people have explored ways to represent a proper fraction as a sum of reciprocals of distinct integers (also known as *unit fractions*). For instance,

$$\frac{7}{15} = \frac{1}{3} + \frac{1}{8} + \frac{1}{120}.$$

The right hand side of the above equation can be written in Hieroglyphics as follows:



See [**11**] for an entertaining and informative introduction to ancient Egyptian mathematics. Even though this idea has been around for at least 3800 years, there are still open problems. Indeed, the Erdős-Straus conjecture (first posed in [**6**]), which is still open at time of writing, asks the question of whether every number of the form $4/n$, with n an integer ≥ 5, can be represented as a sum of 3 or fewer distinct unit fractions.

The breakdown given above for $7/15$ is due to an algorithm by none other than Fibonacci (translated in [**3**]; the algorithm is what Fibonacci calls the "seventh category") and independently by James Joseph Sylvester [**12**]. One can explain it in ancient Egyptian terms in the following way. Say you have 7 loaves of bread that you need to distribute among 15 workers. (Payment in portions of bread loaves is a common trope in primary sources we have access to, such as the Rhind Mathematical Papyrus in the British Museum.) Breaking them in half only gives you 14 pieces, which isn't enough. Instead, break the first *five* loaves into 3 pieces each, making 15 loaves of size $1/3$. Distribute these to your workers, and break up the remaining two loaves into 8 pieces each (since breaking them in 7ths would make 14 slices, which again isn't enough). Distribute 15 of the 16 pieces of size 1/8 to your workers. The remaining 1/8 slice you can break up into 15 pieces (of size $\frac{1}{120}$ each). Each worker gets $\frac{1}{3}$ loaf, then $\frac{1}{8}$ loaf, and finally $\frac{1}{120}$ loaf, verifying the equation above.

One might ask whether a proper fraction may always be broken down into distinct unit fractions. Fibonacci provided an algorithm for doing so, hence the answer is "yes."

Another alternate solution, and one that would not have appeared in ancient papyri as it involves subtraction, is as follows. Break each of your 7 loaves in half, and distribute to 14 of your 15 workers. Each of the 14 workers slices off $\frac{1}{15}$ of their half loaf (i.e., $\frac{1}{30}$ of the original loaf size) to give to the remaining worker, thus sharing the total payment equally. Then everyone has $\frac{7}{15} = \frac{1}{2} - \frac{1}{30}$ loaf. Hence, we have an *alternating* sum of distinct reciprocals.

A parallel problem is to do a similar thing for *rational functions*. That is, given a fraction of the form $f(x)/g(x)$, and assuming it is *proper* (i.e., the degree of the numerator is less than the degree of the denominator), can one represent it as a sum (or alternating sum) of reciprocals of distinct polynomials? For some pairs of polynomials, e.g., if $f(x)$ divides $g(x)$, this is easy. For instance, $\frac{x+3}{x^2+4x+3} = \frac{x+3}{(x+3)(x+1)} = \frac{1}{x+1}$, as in high school Algebra II. What if $f(x)$ does not divide $g(x)$? For example, can one break up $\frac{x^2-x+1}{x^3+2}$ this way? By cleverly using the facts that $x^3+2 = (x^3+1)+1$ and $x^3+1 = (x+1)(x^2-x+1)$, one can proceed as follows:

$$\begin{aligned}\frac{x^2-x+1}{x^3+2} &= \frac{(x+1)(x^2-x+1)}{(x+1)(x^3+2)} = \frac{x^3+1}{(x+1)(x^3+2)} = \frac{(x^3+2)-1}{(x+1)(x^3+2)} \\ &= \frac{x^3+2}{(x+1)(x^3+2)} - \frac{1}{(x+1)(x^3+2)} \\ &= \frac{1}{x+1} - \frac{1}{x^4+x^3+2x+2}.\end{aligned}$$

There is a somewhat general way to do so in the form of the *partial fractions* that come up in integral calculus. For instance, one has

$$\begin{aligned}\frac{x^2}{(x+1)^2(x-4)} &= \frac{9}{25(x+1)} - \frac{1}{5(x+1)^2} + \frac{16}{25(x-4)} \\ &= \frac{1}{\frac{25}{9}(x+1)} - \frac{1}{5(x+1)^2} + \frac{1}{\frac{25}{16}(x-4)}.\end{aligned}$$

However, if the denominator is complicated, the method of partial fractions relies on factoring polynomials and intricate linear algebra. Moreover, if the denominator has irreducible quadratic factors, then unless one expands to the field of complex numbers, this method fails to supply a sum of reciprocals of polynomials. For instance, partial fractions over $\mathbb{R}$ does nothing with the rational function $\frac{2x+1}{x^2+4}$.

In this article, I show that it is always possible to represent a proper rational function as an alternating sum of reciprocals of distinct polynomials, without expanding the base field. Moreover, if one puts further restrictions on the successive denominators of the sum, one can get uniqueness, as shown in the main theorem of this article. In fact, the proof amounts to an algorithm for obtaining the successive polynomials in the denominator, so that no algebraic cleverness is required. This theorem is an analogue of the *Pierce expansion* of any real number in the interval $(0,1)$, a series that is finite precisely when the number is rational. Let me present the following theorem from almost a hundred years ago.[1]

Theorem 1 (Pierce [10]). *Let a, b be integers with $0 < a \leq b$, such that a does not divide b. Then there is a uniquely determined finite sequence $c_0, \ldots, c_n$ of positive integers such that $n \geq 1$, $c_0 < c_1 < \cdots < c_{n-1} < c_n - 1$, and*

$$\frac{a}{b} = \frac{1}{c_0} - \frac{1}{c_0 c_1} + \cdots + \frac{(-1)^n}{c_0 c_1 \cdots c_n} = \sum_{j=0}^{n} \frac{(-1)^j}{\prod_{i=0}^{j} c_i}. \tag{1}$$

Moreover, $c_n \leq b$.

[1]. . . or maybe more? See [**1**, Section 3] for the problem of correct attribution of this theorem.

You can read [**1**] for a delightful exploration of this way to represent rational numbers in the unit interval, including connections with the Monty Hall game show problem and Cantor sets. For a modern proof of the theorem, see [**9**, Theorem 2.1 and Proposition 2.2]. A key ingredient is the successive transformation of improper fractions into quotients and remainders. The Pierce representation of $7/15$ is $\frac{1}{2} - \frac{1}{2\cdot 15} = \frac{1}{2} - \frac{1}{30}$. The proof of the theorem below on rational functions is similar. However, the division algorithm used here is the polynomial algorithm one learns in secondary school algebra, rather than the integer division algorithm learned in primary school.

2. THE MAIN THEOREM. In the following, F is any field, and $F[x]$ is the ring of polynomials in one variable x over F. In thinking through how it all works, one can think of F as $\mathbb{Q}$ (or $\mathbb{R}$), and then $F[x]$ is the set of all polynomials in the variable x with coefficients in $\mathbb{Q}$ (respectively, $\mathbb{R}$).

Main Theorem. *Let F be a field and f, g nonzero polynomials in $F[x]$. Assume that $\deg(f) < \deg(g)$. Then there is a uniquely determined list of nonzero polynomials $h_0, h_1, \ldots, h_n$, such that $n \leq \deg(f)$, $0 < \deg(h_i) < \deg(h_{i+1})$ whenever $0 \leq i < n$, and*

$$\frac{f}{g} = \frac{1}{h_0} - \frac{1}{h_0 h_1} + \cdots + \frac{(-1)^n}{h_0 h_1 \cdots h_n} = \sum_{j=0}^{n} \frac{(-1)^j}{\prod_{i=0}^{j} h_i}. \tag{2}$$

In proving the theorem, I will assume the reader knows how degrees of polynomials behave under addition, subtraction and multiplication. We also need the following well-known result:

Lemma 2 (Polynomial division and remainder). *Let f, g be nonzero polynomials in $F[x]$ with $\deg(f) \leq \deg(g)$. Then there is a unique pair of polynomials q, r with $g = qf + r$, such that $q \neq 0$ and either $r = 0$ or $\deg(r) < \deg(f)$. The polynomials q, r are called the "quotient" and "remainder" of dividing g by f. Write $(q, r) = \mathrm{QR}(g, f)$*

Proof of Main Theorem. **Existence:** If g is a polynomial multiple of f, say $g = hf$, then $\frac{f}{g} = \frac{1}{h}$. Otherwise, Lemma 2 provides polynomials h_0 and r_0 with

$$g = fh_0 + r_0 \quad \text{and} \quad \deg(r_0) < \deg(f). \tag{3}$$

Hence, the leading term of $fh_0 + r_0$ appears in the product fh_0, so we have $\deg(f) + \deg(h_0) = \deg(fh_0) = \deg(g)$. Moreover, dividing Equation (3) through by gh_0, we have

$$\frac{1}{h_0} = \frac{f}{g} + \frac{r_0}{gh_0}, \quad \text{i.e.} \quad \frac{f}{g} = \frac{1}{h_0} - \frac{1}{h_0}\left(\frac{r_0}{g}\right). \tag{4}$$

Note also that $\deg h_0 > 0$, since otherwise we would have $\deg(g) = \deg(f)$, contrary to assumption.

Now act similarly with the rational function r_0/g. If g is a multiple of r_0, say $g = r_0 h_1$, then $r_0/g = 1/h_1$, and so $\frac{f}{g} = \frac{1}{h_0} - \frac{1}{h_0 h_1}$. Otherwise, apply Lemma 2 to get

$$g = r_0 h_1 + r_1 \quad \text{and} \quad \deg(r_1) < \deg(r_0). \tag{5}$$

In either case, we have $\deg(g) = \deg(r_0 h_1) = \deg(r_0) + \deg(h_1)$. Hence, $\deg(h_1) = \deg(g) - \deg(r_0) > \deg(g) - \deg(f) = \deg(h_0)$. And in the latter case, dividing Equation (5) by gh_1 and subtracting, we have

$$\frac{1}{h_1} = \frac{r_0}{g} + \frac{r_1}{gh_1}, \quad \text{i.e.} \quad \frac{r_0}{g} = \frac{1}{h_1} - \frac{1}{h_1}\left(\frac{r_1}{g}\right). \tag{6}$$

Then combining Equations (4) and (6), we obtain

$$\frac{f}{g} = \frac{1}{h_0} - \frac{1}{h_0}\left(\frac{1}{h_1} - \frac{1}{h_1}\left(\frac{r_1}{g}\right)\right) = \frac{1}{h_0} - \frac{1}{h_0 h_1} + \frac{1}{h_0 h_1}\left(\frac{r_1}{g}\right).$$

Continuing inductively, at the kth step when $k \geq 2$, we have by then constructed $r_0, \ldots, r_{k-1}$ and $h_0, \ldots, h_{k-1}$ so that for each $0 < i \leq k-1$ we have $g = r_{i-1}h_i + r_i$, $\deg r_i < \deg(r_{i-1})$, $\deg h_i > \deg h_{i-1}$, and

$$\frac{f}{g} = \frac{1}{h_0} - \frac{1}{h_0 h_1} + \cdots + \frac{(-1)^{k-1}}{h_0 \cdots h_{k-1}} + \frac{(-1)^k}{h_0 \cdots h_{k-1}}\left(\frac{r_{k-1}}{g}\right). \tag{7}$$

Considering the fraction r_{k-1}/g, we find either that g is a multiple of r_{k-1} (so that $g = r_{k-1}h_k$) or we use Lemma 2 to find polynomials h_k and r_k with $g = r_{k-1}h_k + r_k$ and $\deg(r_k) < \deg(r_{k-1})$. In either case, we have $\deg(g) = \deg(r_{k-1}h_k) = \deg(r_{k-1}) + \deg(h_k)$, so that $\deg(h_k) = \deg(g) - \deg(r_{k-1}) > \deg(g) - \deg(r_{k-2}) = \deg(h_{k-1})$. In the first case, we have $\frac{r_{k-1}}{g} = \frac{1}{h_k}$, so that combining with Equation (7), we get

$$\frac{f}{g} = \frac{1}{h_0} - \frac{1}{h_0 h_1} + \cdots + \frac{(-1)^{k-1}}{h_0 \cdots h_{k-1}} + \frac{(-1)^k}{h_0 \cdots h_k}.$$

In the second case, we divide the equation $g = r_{k-1}h_k + r_k$ through by gh_k so that $\frac{r_{k-1}}{g} = \frac{1}{h_k} - \frac{1}{h_k}\left(\frac{r_k}{g}\right)$. Plugging this into Equation (7), we obtain

$$\frac{f}{g} = \frac{1}{h_0} - \frac{1}{h_0 h_1} + \cdots + \frac{(-1)^{k-1}}{h_0 \cdots h_{k-1}} + \frac{(-1)^k}{h_0 \cdots h_k} + \frac{(-1)^{k+1}}{h_0 \cdots h_k}\left(\frac{r_k}{g}\right).$$

Why does this process terminate? Since all degrees are nonnegative integers and

$$\deg(f) > \deg(r_0) > \deg(r_1) > \cdots,$$

we have $\deg(r_0) \leq \deg(f) - 1$, and similarly $\deg(r_1) \leq \deg(r_0) - 1 \leq (\deg(f) - 1) - 1 = \deg(f) - 2$, and so forth. Proceeding inductively, we have for each r_i that $\deg(r_i) \leq \deg(f) - (i+1)$. But also $\deg r_i \geq 0$. Combining the two inequalities, it follows that $\deg(f) - (i+1) \geq 0$, so that $i + 1 \leq \deg(f)$. That is, *there is no* r_i where $i \geq \deg(f)$. Hence, at some step before or at $k = \deg(f)$, the division of g by r_{k-1} has no remainder. That is, $n \leq \deg(f)$ in (2).

Uniqueness: Suppose $h_0, \ldots, h_n$ are polynomials such that $0 < \deg h_0 < \deg h_1 < \cdots < \deg h_n$ and Equation (2) holds. We inductively define polynomials $r_0, \ldots, r_{n-1}$ via $r_0 := g - h_0 f$ and for $1 \leq i \leq n$ set $r_i := g - h_i r_{i-1}$.

First, note that by the equations defining the r_i, combined with Equation (2), it follows that $r_n = 0$, so that $g = h_n r_{n-1}$. See Proposition 3 below. In particular, $\deg g = \deg(h_n) + \deg r_{n-1}$.

Second, I claim that for all $0 \leq i < n$, we have $\deg(r_i) < \deg g$. Let us use descending induction starting with $i = n-1$. When $i = n-1$ we have $g = h_n r_{n-1}$ so that $\deg g = \deg h_n + \deg r_{n-1} > \deg r_{n-1}$. So suppose $i \geq 1$ and $\deg g > \deg r_i$. Then since $g = h_i r_{i-1} + r_i$, it follows that $\deg g = \deg(h_i r_{i-1}) = \deg(h_i) + \deg(r_{i-1}) > \deg r_{i-1}$. Thus it also follows that for each $0 \leq i < n-1$, we have $\deg(g) = \deg(h_{i+1} r_i)$ (since $g = h_{i+1} r_i + r_{i+1}$ and $\deg g > \deg r_{i+1}$), and also $\deg g = \deg(h_0 f)$ (since $g = h_0 f + r_0$ and $\deg g > \deg r_0$).

Third, we show that for each $0 < i < n$, we have $\deg(r_i) < \deg r_{i-1}$, and also $\deg r_0 < \deg f$. To see this, note that for each $0 < i < n$ we have $\deg g = \deg h_{i+1} + \deg r_i = \deg h_i + \deg r_{i-1}$, so that since $\deg h_i < \deg h_{i+1}$ we have $0 < \deg h_{i+1} - \deg h_i = \deg r_{i-1} - \deg r_i$, whence $\deg r_i < \deg r_{i-1}$. Similiarly, $\deg g = \deg h_1 + \deg r_0 = \deg h_0 + \deg f$, so that since $\deg h_0 < \deg h_1$ we have $0 < \deg h_1 - \deg h_0 = \deg f - \deg r_0$, whence $\deg r_0 < \deg f$.

It follows then that $(h_0, r_0) = \mathrm{QR}(g, f)$, and for each $0 < i < n$ we have $(h_i, r_i) = \mathrm{QR}(g, r_{i-1})$, and $h_n = g/r_{n-1}$. Hence by Lemma 2, the list $h_0, \ldots, h_n$ is uniquely determined from the quotient-remainder algorithm, iteratively applied. ■

The bit of algebra that remains is below.

Proposition 3. *Let F be a field. Let $f, g, h_0, \ldots, h_n$ be nonzero elements of $F[x]$ that satisfy Equation (2). Set $r_0 := g - h_0 f$, and for $1 \leq i \leq n$, inductively set $r_i := g - h_i r_{i-1}$. Then $r_n = 0$.*

Proof. Multiplying Equation 2 by $g h_0 \cdots h_n$, we have

$$h_0 h_1 \cdots h_n f = g \cdot \left(\sum_{i=0}^{n} (-1)^i \prod_{j=i+1}^{n} h_j \right). \tag{8}$$

Let us first show by induction that for $0 \leq t \leq n-1$, we have $r_n = g \cdot (1 + \sum_{j=0}^{t-1} (-1)^{j+1} \prod_{k=0}^{j} h_{n-k}) + (-1)^{t+1} r_{n-t-1} \prod_{k=0}^{t} h_{n-k}$. When $t = 0$, this is true because $r_n = g - r_{n-1} h_n$. So suppose it is true for some t with $0 \leq t < n-1$. Then $r_n = g \cdot (1 + \sum_{j=0}^{t-1} (-1)^{j+1} \prod_{k=0}^{j} h_{n-k}) + (-1)^{t+1} r_{n-t-1} \prod_{k=0}^{t} h_{n-k} = g \cdot (1 + \sum_{j=0}^{t-1} (-1)^{j+1} \prod_{k=0}^{j} h_{n-k}) + (-1)^{t+1} (g - r_{n-t-2} h_{n-t-1}) \prod_{k=0}^{t} h_{n-k} = g \cdot (1 + \sum_{j=0}^{t} (-1)^{j+1} \prod_{k=0}^{j} h_{n-k}) + (-1)^{t+2} r_{n-t} \prod_{k=0}^{t+1} h_{n-k}$, completing the inductive step.

In particular, setting $t = n-1$, we have

$$\begin{aligned}
r_n &= g \cdot \left(1 + \sum_{j=0}^{n-2} (-1)^{j+1} \prod_{k=0}^{j} h_{n-k} \right) + (-1)^n r_0 \prod_{k=0}^{n-1} h_{n-k} \\
&= g \cdot \left(1 + \sum_{j=0}^{n-2} (-1)^{j+1} \prod_{k=0}^{j} h_{n-k} \right) + (-1)^n (g - h_0 f) \prod_{k=0}^{n-1} h_{n-k} \\
&= g \cdot \left(1 + \sum_{j=0}^{n-1} (-1)^{j+1} \prod_{k=0}^{j} h_{n-k} \right) + (-1)^{n+1} f \prod_{k=0}^{n} h_{n-k} \\
&= (-1)^n \cdot (g \left(\sum_{i=0}^{n} (-1)^i \prod_{j=i+1}^{n} h_j \right) - h_0 h_1 \cdots h_n f),
\end{aligned}$$

which equals zero because of Equation (8). ■

Example 4. To see how the above algorithm works, let us consider the problem of representing the fraction $\frac{x^2+5x+1}{x^4+116x+25}$ as such an alternating sum. That is, $f = x^2 + 5x + 1$ and $g = x^4 + 116x + 25$. Using polynomial division and remainder, one finds that $x^4 + 116x + 25 = (x^2 - 5x + 24)(x^2 + 5x + 1) + (x + 1)$. That is $h_0 = x^2 - 5x + 24$ and $r_0 = x + 1$. Now divide $g = x^4 + 116x + 25$ by $r_0 = x + 1$ to get $x^4 + 116x + 25 = (x + 1)(x^3 - x^2 + x + 115) - 90$. That is, $h_1 = x^3 - x^2 + x + 115$ and $r_1 = -90$. Finally, the polynomial division algorithm of $g = x^4 + 116x + 25$ by $r_1 = -90$ just divides each of the coefficients by -90 and leaves no remainder. That is, $h_2 = (-1/90)x^4 - (116/90)x - 5/18$, and there is no r_2. Thus, we have

$$\begin{aligned}\frac{x^2+5x+1}{x^4+116x+25} =& \frac{1}{x^2-5x+24} - \frac{1}{(x^2-5x+24)(x^3-x^2+x+115)} \\ &+ \frac{1}{(x^2-5x+24)(x^3-x^2+x+115)\left(\frac{-1}{90}x^4 - \frac{116}{90}x - \frac{5}{18}\right)}.\end{aligned}$$

I posit that it would be hard to come up with such a breakdown without an algorithm such as the one given here.

3. A NON-ALTERNATING VARIANT. A natural question is: can one replace the *alternating sum* in the Main Theorem with an ordinary (i.e. without powers of negative 1) sum? The answer is yes. Simply set $p_0 = h_0$ and $p_i := -h_i$ for $1 \le i \le n$. Then we have $h_0 h_1 \cdots h_i = p_0(-p_1)\cdots(-p_i) = (-1)^i p_0 p_1 \cdots p_i$, and then Equation (2) becomes

$$\frac{f}{g} = \frac{1}{p_0} + \frac{1}{p_0 p_1} + \cdots + \frac{1}{p_0 p_1 \cdots p_n}. \tag{9}$$

In this representation, all the conditions on degrees are the same as in the Main Theorem, and we get uniqueness for the same reason as well. That is, we have the following result:

Theorem 5. *Let F be a field and f, g nonzero polynomials in $F[x]$ such that $\deg(f) < \deg(g)$. Then there is a uniquely determined list of nonzero polynomials $p_0, p_1, \ldots, p_n$ such that $n \le \deg(f)$, $0 < \deg(p_i) < \deg(p_{i+1})$ whenever $0 \le i < n$, and Equation (9) holds.*

This then is a polynomial version of the *Engel expansion* of a rational number in the unit interval:

Theorem 6 (Engel [4]; Cohen [2]). *Let a, b be integers with $0 < a < b$. Then there is a uniquely determined finite sequence $d_0, \ldots, d_n$ of positive integers, such that $n \ge 0$, $2 \le d_0 \le d_1 \le \cdots \le d_n$ and*

$$\frac{a}{b} = \frac{1}{d_0} + \frac{1}{d_0 d_1} + \cdots + \frac{1}{d_0 d_1 \cdots d_n}.$$

Moreover, $d_n \le b$.

For a modern proof of the above theorem, see [**8**]. Note the differences in the assumptions on the sequences of the integers d_i in this theorem from the integers c_i in Theorem 1. Note also that the representation given in the Introduction of $7/15$ as $\frac{1}{3} + \frac{1}{8} + \frac{1}{120}$ does *not* satisfy the Engel-Cohen theorem, as 8 is not a multiple of 3.

In fact the (Engel-)Cohen representation of $\frac{7}{15}$ is $\frac{1}{3} + \frac{1}{3\cdot 3} + \frac{1}{3\cdot 3\cdot 5} = \frac{1}{3} + \frac{1}{9} + \frac{1}{45}$. The denominators here are quite different also from the Pierce representation $\frac{1}{2} - \frac{1}{30}$. This is typical, in that the Cohen, Fibonacci, and Pierce representations of a proper fraction involve really different denominators in general. By contrast, the representations of rational functions in the theorems given here vary only in signs.

4. CONSTRUCTING REAL NUMBERS AND INVERSE POWER SERIES. Pierce's theorem and Engel's theorem are usually seen not as ways to represent *rational* numbers, but as special cases of ways to represent *real* numbers. Indeed, Pierce and Engel showed that sequences of the types given in the theorems above, if extended to infinity, can be used as unique representations of all real numbers in the (open) unit interval.

Similarly, the Main Theorem can be seen as a special case of a way to represent *inverse power series*, which in turn are a generalization of rational functions, in the same sort of way that real numbers are a generalization of rational numbers. Such a viewpoint is exploited in [**7**], and the perspective in that article might also yield a proof of the main theorem here. But the proof given in this article is more elementary, as unlike the authors in [**7**], I have not resorted to valuation theory.

5. EUCLIDEAN DOMAINS. In [**5**], a different approach is taken. As one learns in a graduate (or advanced undergraduate) algebra course, the rings $\mathbb{Z}$ and $F[x]$ are special cases of *Euclidean domains*, in the sense that there is a function $\varphi : D \to \mathbb{N}$ (where D is the Euclidean domain), called a *Euclidean function*, such that some analogue of Lemma 2 holds. When $D = F[x]$, $\varphi =$ degree, whereas when $D = \mathbb{Z}$, $\varphi =$ the absolute value function. It is shown in [**5**] that when D is a Euclidean domain with Euclidean function φ, if $f, g \in D$ are nonzero with $\varphi(f) \leq \varphi(g)$, then $\frac{f}{g}$ can be written as a sum of distinct unit fractions, with denominators from D. This is in some sense a generalization of the Main Theorem above, as well as the original Egyptian fraction theorem proved by Fibonacci, except that in the main theorem of [**5**], there is much less control over the denominators in question than one sees here in the polynomial case.

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