An analysis of the renormalization group method for asymptotic expansions with logarithmic switchback terms

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Abstract

The renormalization group method of Chen, Goldenfeld, and Oono offers a comprehensive approach to formally computing asymptotic expansions of the solutions to singular perturbation problems and multi-scale problems. In particular, the RG method applies to a broad array of problems customarily treated with disparate methods, such as the method of multiple scales, boundary layer theory, matched asymptotic expansions, Poincare-Lindstedt theory, the WKBJ method with and without turning points, the method of averaging, and others. For problems in which the expansions are in powers of the small parameter, it has been shown that the RG method leads to uniformly valid asymptotic expansions of the solutions. In addition, it has been shown that the RG condition constitutes an invariance condition, that the RG method is effectively a resummation technique, and that it is equivalent to normal form theory for certain broad classes of perturbed ordinary differential equations. However, there has not yet been an analysis of the validity of the RG method, *i.e.,* a demonstration that it leads to uniformly valid asymptotic expansions, for problems in which the expansions also involve logarithms of the small parameter as gauge functions. The aim of this article is to provide this justification of the RG method. In particular, we extend the approach developed in DeVille, *et al.* [*Physica D* 237 no. 8, 1029 -1052] from the classes of autonomous and non-autonomous perturbations considered there to include the non-autonomous systems subject to singular perturbations for which the solutions involve logarithmic gauge functions. This framework is built upon the relationship between the RG method and normal form theory. We apply the RG method to three successively-more complex examples and also elucidate the common general features.

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1 Introduction

Logarithmic switchback terms arise in the solutions of a diverse array of differential equations, including the Navier-Stokes equations for low Reynolds number flow past a cylinder and past a sphere [17, 18, 33, 39], the model for a meniscus on a needle [26], the $\log(\epsilon)$ boundary layer problem of Bender and Orszag see Chapter 9.4 of [1], the propagation speed of pulled fronts in reaction-diffusion equations with cut-offs [8], the Lagerstrom model [6, 9, 11, 15, 16, 20, 27, 28, 34, 35, 37], to name a few. The central feature common to these equations is that the asymptotic expansions of the solutions contain not just powers of the small parameter but also terms involving the logarithm, or even nested logarithms, of the small parameter. A canonical example of a perturbation expansion with a logarithmic switchback term is the expansion,

$$u(t,\epsilon) = u_0(t) + \epsilon\log(\epsilon)\tilde{u}(t) + \epsilon u_1(t) + \cdots, \tag{1.1}$$

where $0 < \epsilon \ll 1$ is a small parameter, t is the independent variable and u is the dependent variable.

A primary, though not exclusive, method for finding the logarithmic terms is the method of matched asymptotic expansions. Indeed, we emphasize from the outset that one may find terms dependent on the logarithm of the small parameter via other methods, as has been shown in [8, 11, 12, 31, 32], for example. In the method of matched asymptotic expansions, one posits an outer expansion, which is a valid asymptotic expansion of the solution to the desired order in the small parameter over a large part of the domain of interest, as well as an inner expansion, which is valid in a narrower layer, typically a boundary layer. Then, one matches the two expansions in the overlap domain in which they are both valid, obtaining a uniformly valid asymptotic expansion. However, in the process of matching for these problems, one finds that it is necessary to insert a term proportional to the logarithm of the small parameter in one of the expansions in order to match all of the terms. Hence, the label 'switchback' term. For example, a term of $\mathcal{O}(\epsilon\log(\epsilon))$ sits in between terms of $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$, asymptotically as $\epsilon \to 0$, and if it is not in the original expansions one must go back and insert it, as shown in (1.1).

Since the time of the earliest analyses of these types of problems, such as for example [17, 18, 33], asymptotic expansions with logarithmic terms have been a standard component of graduate-level courses in asymptotic analysis, perturbation methods, and applied mathematics generally. We refer the reader to, among others, [1, 9, 13, 14, 19, 24, 29, 36, 38] for some of the standard textbooks. In addition, an informative case study of logarithmic switchback terms in perturbation expansions is presented in [25]. There, a model problem was used to demonstrate that terms involving the logarithm of the small parameter arise naturally due to $\log(x)$ terms in the expansions of the solutions, which in turn exist due to the presence of irregular singular points in the underlying linear parts of the equation. This is the case in other examples, as well, that a logarithmic gauge function arises when one of the expansions depends at some order on the logarithm of the independent variable.

Recently, it has been shown that the renormalization group (RG) method of Chen, Goldenfeld, and Oono can directly yield expansions with logarithm terms, without any need of foreknowledge of their presence or of matching, see [2] and Section IV of [3]. The RG procedure begins with the naive perturbation expansion in powers of the small parameter. This expansion is renormalized, by applying a near-identity coordinate change that removes the non-resonant terms. Next, the renormalization group condition is applied to the renormalized expansion, yielding the RG, or amplitude, equations. Finally, the RG equations are solved and this solution is substituted back into the renormalized expansion to generate the final, uniformly-valid asymptotic expansion. For some problems with logarithmic switchback terms, it was shown in [2, 3] that the RG method naturally identifies the $\log(\epsilon)$ terms and that the final RG expansion agrees, to the orders considered there, with the known matched asymptotic results. Indeed, this is one of the strengths of the RG method, that one does not need foreknowledge, artistry, or matching to obtain terms with different functional dependence on the small parameter, including logarithmic terms. We refer the reader to the following incomplete list of articles [2, 3, 4, 5, 7, 10, 21, 22, 23, 30, 41, 42] for applications and analyses of the Chen, Goldenfeld, and Oono RG method to other types of perturbation problems.

In this article, we analyze how the RG method generates the uniformly valid asymptotic expansions complete with the logarithmic terms. This important class of perturbation problems has not yet been considered in any of the existing analyses. Specifically, we show how the RG method naturally induces a coordinate change that puts a perturbed vector field into normal form, and we show that the normal form is equivalent to the reduced, or amplitude, equations. This equivalence is then used to rigorously establish the uniform validity of the renormalized expansions, thus justifying the RG procedure for this class of perturbation problems.

This three-part analysis of problems with logarithmic switchback terms is a natural extension of our earlier work in [7] and of the analysis of the RG method first presented in [42]. Two large classes of singularly-perturbed ordinary differential equations, with autonomous and non-autonomous perturbation terms respectively were studied in [7]. In particular, we showed that for systems with autonomous perturbations, the reduced or amplitude equations generated by the RG method are equivalent to the classical Poincare-Birkhoff normal forms. In addition, for the class of problems with non-autonomous perturbations, it was shown that the RG-generated amplitude equations constitute time-asymptotic normal forms, which are based on Krylov-Bogoliubov-Mitropolskii (KBM) averages.

For both classes of problems considered in [7], it was also shown that the coordinate change used in the RG approach is equivalent, up to translation between the spaces of solutions and vector fields, to the coordinate change used in the normal form theory. The equivalence of these coordinate changes, modulo the spaces in which they are defined, shows that there is a commuting diagram, as follows. Starting from the original vector field, one may directly apply a near-identity coordinate change, labeled NF, to derive the normal form. Alternatively, one may carry out the three-step RG procedure, which begins with the derivation, labeled RG_1, of the naive perturbation expansion, makes a coordinate change, RG_2, in order to renormalize the expansion, and then applies the RG condition, RG_3, to transform the renormalized expansion into the reduced, or amplitude, equation. This commutation of operations is illustrated in the following figure:

$$\begin{array}{ccc} S & \xrightarrow{\mathsf{RG}_2} & S \\ {\scriptstyle \mathsf{RG}_1}\big\uparrow & & \big\downarrow{\scriptstyle \mathsf{RG}_3} \\ V & \xrightarrow{\mathsf{NF}} & V. \end{array} \tag{1.2}$$

The essential reductive step is the near-identity change of coordinates, RG_2. This change of coordinates removes non-resonant terms from the asymptotic expansion, in the same manner that the normal form coordinate change removes non-resonant terms from the vector field.

This relationship between the RG method and asymptotic normal form theory suggests that, see [7], the RG method offers a new approach for deriving normal forms for differential equations, including for some systems for which there is as yet no normal form theory. The advantage offered by the RG approach is that one can typically more readily identify resonant terms from naive perturbation expansions than from the non-autonomous vector fields themselves.

In this article, we show that the framework provided by this commuting diagram extends naturally from the classes of problems considered in [7], in which switchback terms cannot arise, to the perturbation problems considered here in which logarithmic switchback terms do arise. We show that the RG method derives the key reductive, near-identity coordinate changes in a straightforward fashion also for these problems. Also, we show that it yields the normal forms for these non-autonomous systems and that the uniform validity of the asymptotic expansions it generates can be established using this property, thus fully extending the results of [7].

We achieve the main goals of this article by examining a series of three examples, chosen to illustrate the key features. First, we illustrate the RG method through an application to the explicitly solvable boundary value problem,

$$(x+\epsilon)y'' + y' = 1, \tag{1.3}$$

with boundary conditions $y(0) = 0$ and $y(1) = 2$. This equation was introduced by Eckhaus, see Section 2.3 of [24], as a vehicle to illustrate some of the features of asymptotic matching in the presence of logarithmic switchback terms. It turns that, for this equation, all of the terms at $\mathcal{O}(\epsilon^2)$ and higher order vanish identically and that the expansion with just lower order terms includes a $\log(\epsilon)$ term and represents an exact solution. The simplicity of this example makes it an ideal problem on which to illustrate the application of the RG method. We show that the RG method naturally identifies the exact solution, including the $\log(\epsilon)$ term. We also observe that the fact that the higher order terms all vanish does not influence the application of the RG method at the lower orders, nor is any advantage gained from the presence of the exact solution.

Next, we consider the Lagerstrom model problem:

$$u'' + \frac{n-1}{x}u' + uu' = 0, \tag{1.4}$$

with $n = 3$ and for $u(x)$ satisfying the boundary conditions $u(x = \epsilon) = 0$ and $\lim_{x\to\infty} u(x) = 1$. This model problem was developed by Paco Lagerstrom as a vastly-simplified caricature of the PDE problems of low Reynolds number fluid flow past a sphere ($n = 3$) and past a cylinder ($n = 2$). The uniformly valid solution over $[\epsilon, \infty)$ involves terms proportional to $\log(\epsilon)$. The full history, including the more-complex model system in which this equation is embedded, is given in [6, 24]. We also observe that analysis has been carried out for this model establishing the existence of solutions to this BVP, the accuracy of the asymptotic expansions, and their uniform validity. See [9, 11, 12, 15, 16, 20, 24, 25, 27, 28, 31, 32, 34, 35, 37].

Our main results for the Lagerstrom model are designed to complement the existing analyses of this model, not to recover the known results, by focusing on application of the RG method to this model and on establishing the equivalence between the RG and normal form methods. Here, we focus on the essential form given by (1.4), following [25]. Our main results for this second example include a demonstration that the RG method yields the logarithmic term, identification of the fact that the coordinate change generated by the RG method in the space of solutions of the equation may be used to find the coordinate change in the space of the vector field to put the original system (1.4) into its normal form, and a demonstration that the normal form is equivalent to the renormalized equation, thus also enabling us to establish the validity of the renormalized expansion.

The third, and final, example we consider is a model problem from Bender and Orszag,

$$\epsilon y'' + xy' - xy = 0, \tag{1.5}$$

with $y(x_0) = A_0$ and $y'(x_0) = B_0$. Here, A_0 and B_0 are given constants, and x_0 denotes a given initial point, see Chapter 9.4 of [1]. In this example, the asymptotically small parameter is $\sqrt{\epsilon}$, and the logarithmic term is of the form $\epsilon \log(\epsilon)$. Hence, the switchback occurs between the first-order and second-order terms. Despite this fundamental difference with the other two examples, the RG method nevertheless naturally identifies the $\log(\epsilon)$ term at the appropriate order and generates a uniformly valid asymptotic expansion. In addition, this example does not share the specialized feature that the Lagerstrom model possesses, namely that the outer expansion is a uniformly valid expansion. Hence, this third example also permits us to establish that it is the key features, and not any specialized properties, that lead to the efficacy of the RG approach. Finally, we also derive the normal form for this system, and use that to justify the validity of the renormalized expansion.

The phenomenon of logarithmic switchback has recently also been studied using geometric desingularization (also known as the blow-up method) from the fields of algebraic geometry and dynamical systems, see [31, 32]. This method desingularizes the vector field in a neighborhood of a degenerate equilibrium. In particular, the degenerate equilibrium is blown-up into a (topological) hemisphere. For the model studied in [31, 32], the flow on this hemisphere, which may be analyzed using classical invariant manifold theory, reveals that the terms proportional to the logarithm of the small parameter arise due to a resonance among the eigenvalues of a particular fixed point on the equator of the hemisphere, and the associated relatively long passage through the neighborhood of this fixed point.

This article is organized as follows: in section 2, we present the analysis of the Eckhaus example. In section 3, we present the analysis of the Lagerstrom model. The phenomenon of higher-order switchback in the Bender-Orszag example is analyzed in section 4. The article concludes in Section 5 with a brief general analysis of the relationship between the RG method and normal form theory, including the importance of the second step in the RG method, RG_2, as it relates to isolating the resonant, logarithmic terms. We have relegated a number of the more involved calculation to Appendices.

2 Application of the RG method to the Eckhaus example (1.3)

In this section, we apply the RG method to the Eckhaus example (1.3). As stated above, we begin with this example because it offers an elementary and lucid framework on which to illustrate the method. In general, the RG method consists of the following four steps:

- RG_1: The derivation of a naive perturbation expansion in powers of the small parameter, valid for arbitrary initial time t_0.
- RG_2: The application of a near-identity coordinate change applied to the initial condition that removes the non-resonant and sub-resonant terms from the naive expansion. The new expansion is labeled the renormalized expansion.
- RG_3: The application of the RG condition – that the partial derivative of the renormalized expansion with respect to t_0 vanishes for all t_0 – to the renormalized expansion to generate the RG, or amplitude, equations. These equations are equivalent to the normal form equations, see (1.2).
- Finally, the RG equations are solved. The solutions are placed in the renormalized expansion. The initial time t_0 was arbitrary and by selected $t_0 = t$ the secular terms in the renormalized expansion are removed and a uniformly valid asymptotic approximation is recovered.

We now turn our attention to the application of the RG procedure to equation (1.3),

$$(x+\epsilon)y'' + y' = 1, \tag{2.1}$$

with $y(0) = 0$ and $y(1) = 2$. The exact solution to this boundary value problem is

$$y(x;\epsilon) = x - \frac{\log(1+\frac{x}{\epsilon})}{\log\frac{\epsilon}{1+\epsilon}}, \tag{2.2}$$

see Section 2.3 of [24]. We will show that the RG method directly yields this exact solution, and hence that it also naturally identifies the $\log(\epsilon)$ term in the expansion without any foreknowledge thereof or of matching.

2.1 RG_1: derivation of a naive perturbation expansion.

This equation has a boundary layer near $x = 0$ of thickness of $\mathcal{O}(\epsilon)$. Rescaling $t = \frac{x}{\epsilon}$, we find an equivalent boundary value problem,

$$(t+1)\ddot{y} + \dot{y} = \epsilon, \quad y_0(0) = 0,\ y_0(\frac{1}{\epsilon}) = 2. \tag{2.3}$$

This formulation is known as the inner equation. To begin, one supposes a naive perturbation expansion of the form,

$$y(t,\epsilon) = y_0(t) + \epsilon y_1(t) + \mathcal{O}(\epsilon^2).$$

Substituting this ansatz into (2.3), we solve order by order in ϵ. To leading order, we find

$$(1+t)\ddot{y}_0 + \dot{y}_0 = 0.$$

To proceed in as general of a manner as possible, we expand this as a system of first order equations,

$$\begin{aligned}\dot{y}_0 &= z_0\\ \dot{z}_0 &= -\frac{z_0}{1+t}.\end{aligned}$$

We solve this system as an initial value problem with arbitrary initial values $y_0(t_0)$ and $z_0(t_0)$. The leading order solution is

$$y_0(t) = y_0(t_0) + z_0(t_0)(1+t_0)\log\frac{1+t}{1+t_0}, \quad z_0(t) = z_0(t_0)\frac{1+t_0}{1+t}. \tag{2.4}$$

Next, we collect terms of $\mathcal{O}(\epsilon)$,

$$\begin{aligned}\dot{y}_1 &= z_1\\ \dot{z}_1 &= -\frac{z_1}{1+t} + \frac{1}{1+t}.\end{aligned}$$

In contrast with the leading order system, here we impose zero initial conditions arguing that if the system possessed some initial conditions at this order they could be readily absorbed into the leading order (arbitrary) initial data. The solution to this initial value problem can then be deduced (we omit the details) and the naive perturbation expansion is computed to be

$$\begin{aligned}y(t,\epsilon) &= y_0(t_0) + z_0(t_0)(1+t_0)\log\left(\frac{1+t}{1+t_0}\right) + \epsilon\left((t-t_0) - (1+t_0)\log\left(\frac{1+t}{1+t_0}\right)\right)\\ z(t,\epsilon) &= z_0(t_0)\frac{1+t_0}{1+t} + \epsilon\frac{t-t_0}{1+t}.\end{aligned} \tag{2.5}$$

There are no higher order terms, and the formulas in (2.5) are exact. This follows from the fact that $y_n \equiv 0$ for all $n \geq 2$, since the governing equations at $\mathcal{O}(\epsilon^n)$ are

$$\dot{y}_n = z_n, \quad \dot{z}_n = -\frac{z_n}{1+t},$$

with zero initial conditions.

2.2 RG_2: renormalization of the naive perturbation expansion (2.5)

Since we are supposing an asymptotic expansion of the solution, it is important that the naive expansion retains its asymptotic nature on the interval of interest (here that interval is $\mathcal{O}(\epsilon^{-1})$). When it does not do so, one says that there is a secularity in the problem. Under conventional asymptotic analysis, this was an indication that the naive perturbation expansion was an incorrect starting point for analysis, and the approach would be abandoned in favor of some other asymptotic method. The power of the RG method is to deal with the secularities present in the naive expansion.

Examining (2.5), we observe the existence of secularities in both the equations for $y_0(t)$ and $z_0(t)$. The terms proportional to $t - t_0$ diverge and one finds that on the asymptotically large interval of interest the naive expansion becomes disordered. Following the RG procedure, we introduce a near-identity change of coordinates that replaces the initial conditions $y_0(t_0)$ and $z_0(t_0)$ with new constants of integration, i.e. the Cauchy data is renormalized. In particular, we substitute

$$y_0(t_0) = \sum_{j=0}^{\infty} \epsilon^j Y_j(t_0), \quad z_0(t_0) = \sum_{j=0}^{\infty} \epsilon^j Z_j(t_0), \tag{2.6}$$

into the naive perturbation expansion, (2.5). This gives

$$\begin{aligned} y(t,\epsilon) &= Y_0(t_0) + Z_0(t_0)(1+t_0)\log\frac{1+t}{1+t_0} \\ &+ \epsilon\left(Y_1(t_0) + Z_1(t_0)(1+t_0)\log\frac{1+t}{1+t_0} + (t-t_0) - (1+t_0)\log\frac{1+t}{1+t_0}\right) + \mathcal{O}(\epsilon^2) \\ z(t,\epsilon) &= Z_0(t_0)\frac{1+t_0}{1+t} + \epsilon\left(Z_1(t_0)\frac{1+t_0}{1+t} + \frac{t-t_0}{1+t}\right) + \mathcal{O}(\epsilon^2). \end{aligned}$$

Now $Y_1(t_0)$ and $Z_1(t_0)$ are chosen so that all instances of t_0 that appear in non-resonant terms are removed. We observe that in the expansion for $y(t,\epsilon)$, the only non-resonant term at $\mathcal{O}(\epsilon)$ is the final term $(1+t_0)\log\frac{1+t}{1+t_0}$. This term is removed by the change of coordinates,

$$Y_1(t_0) = 0, \quad Z_1(t_0) = 1. \tag{2.7}$$

Proceeding to higher order, one finds $Y_j(t_0) \equiv 0$ and $Z_j(t_0) \equiv 0$ for all $j \geq 2$. Therefore, we have derived the renormalized expansion,

$$\begin{aligned} y(t,\epsilon) &= Y_0(t_0) + Z_0(t_0)(1+t_0)\log\frac{1+t}{1+t_0} + \epsilon(t-t_0) \\ z(t,\epsilon) &= Z_0(t_0)\frac{1+t_0}{1+t} + \epsilon. \end{aligned} \tag{2.8}$$

2.3 RG_3: Applying the RG condition to (2.8)

With the renormalized expansion in hand, we now apply the RG condition. As noted above, the RG condition is an invariance condition that insures that the integration constants $Y_0(t_0)$ and $Z_0(t_0)$ depend on t_0 in such a way that the solution remains invariant as t_0 changes. That is, the RG condition is

$$\frac{\partial}{\partial t_0}\begin{pmatrix} y(t,\epsilon) \\ z(t,\epsilon) \end{pmatrix} = 0.$$

Applying the RG condition to (2.8) and simplifying, we find the following evolution equations for $Y_0(t_0)$ and $Z_0(t_0)$,

$$\begin{aligned} \frac{dY_0}{dt_0} &= Z_0 + \epsilon \\ \frac{dZ_0}{dt_0} &= -\frac{Z_0}{1+t_0}. \end{aligned} \tag{2.9}$$

These are the RG equations, also referred to as amplitude equations. They contain only resonant terms and represent the simplest form in which system (2.3) can be put. The near-identity coordinate change (2.6), with Y_1 and Z_1 given by (2.7), is the key step that removes the non-resonant term, generating the renormalized expansion. We remark that these amplitude equations could also be derived by setting $\dot{y} = z$ and then rescaling $z = Z + \epsilon$ and $y = Y$.

The solutions to the RG, or amplitude equations, (2.9) are

$$Y_0(t_0) = C_1 \log(1 + t_0) + C_2 + \epsilon t_0, \quad Z_0(t_0) = \frac{C_1}{1 + t_0}.$$

These solutions are substituted back into (2.8), which gives

$$\begin{aligned} y(t, \epsilon) &= C_1 \log(1 + t_0) + C_2 + C_1 \log \frac{1 + t}{1 + t_0} + \epsilon t \\ z(t, \epsilon) &= C_1 \frac{1}{1 + t} + \epsilon. \end{aligned}$$

The RG condition ensures that this provides the same solution for any value of t_0 and so selecting $t_0 = t$ we arrive at the final expansion,

$$y(t, \epsilon) = C_1 \log(1 + t) + C_2 + \epsilon t.$$

Finally, we select C_1 and C_2 so that the boundary conditions are satisfied. We find that $C_1 = \frac{1}{\log(1+\frac{1}{\epsilon})}$, $C_2 = 0$. Hence, the final expansion is

$$y(t, \epsilon) = \frac{\log(1 + t)}{\log(1 + \frac{1}{\epsilon})} + \epsilon t. \tag{2.10}$$

Therefore, by rescaling $x = \epsilon t$, we see that the RG method has recovered the exact solution (2.2). Also, in the course of this derivation, we see that it identifies the logarithmic gauge function, without any *a priori* knowledge of this term. The logarithmic term at $\mathcal{O}(1)$ in (2.5) is responsible for this switchback term.

3 The Lagerstrom Model Equation

In this section, we study the Lagerstrom model equation,

$$u'' + \frac{n - 1}{x} u' + u u' = 0, \tag{3.1}$$

with boundary conditions given by

$$\begin{aligned} u(x = \epsilon) &= 0 \\ u(x = \infty) &= 1. \end{aligned} \tag{3.2}$$

We will focus on the case $n = 3$. In this case, the Lagerstrom model was developed as a caricature of small Reynolds number flow past a sphere. The dependent variable u may be thought of as the fluid velocity, which is uniform far from the sphere and which is zero on the surface of a sphere with radius ϵ. This is a prototypical boundary layer problem, where the velocity field is nearly uniform everywhere except close to the surface of the sphere.

3.1 Uniformly valid solution: brief review of matched asymptotic expansions

In this section, we briefly review the construction via matched asymptotic expansions of the uniformly valid solution of (3.1) and (3.2). With this method, one constructs an inner asymptotic expansion, valid in the boundary layer near $x = \epsilon$, and an outer asymptotic expansion valid outside the boundary layer all the way out to ∞, and then one matches the results in a domain in which both expansions are valid. Introducing the boundary layer variable $t = x/\epsilon$ and recycling the prime to denote the derivative with respect to t, we find the inner equation

$$u'' + \frac{n - 1}{t} u' + \epsilon u u' = 0 \tag{3.3}$$

with new boundary conditions

$$\begin{aligned} u(t=1) &= 0 \\ u(t=\infty) &= 1. \end{aligned}$$

The inner expansion is of the form,

$$u(t,\epsilon) = u_0(t) + \epsilon \log \epsilon \; \tilde{u}(t) + \epsilon u_1(t) + \dots .$$

One solves the inner equation (3.3) order by order to obtain

$$\begin{aligned} u_0(t) &= (1 - \frac{1}{t}) \\ \tilde{u}(t) &= \tilde{A}(1 - \frac{1}{t}) \\ u_1(t) &= -\log(t) - \frac{\log(t)}{t} + A(1 - \frac{1}{t}). \end{aligned}$$

Observe that no choice of A exists to make the inner expansion uniformly valid; due to the logarithmic divergence of u_1. This problem is dealt with by matching below.

Next, one supposes a naive perturbation expansion for the outer equation

$$u = U_0(x) + \epsilon U_1(x) + \dots .$$

The flow is uniform at infinity. Hence, $U_0 = 1$, and then at $\mathcal{O}(\epsilon)$ one finds

$$U_1(x) = B \int_x^\infty \frac{e^{-\rho}}{\rho^2} d\rho.$$

The integral here is the exponential integral e_2,

$$e_n(x) = \int_x^\infty \frac{e^{-\rho}}{\rho^n} d\rho.$$

Asymptotically, for small x, $e_2(x) \sim \frac{1}{x} + \log(x) + \gamma - 1 + \dots$, where γ is the Euler constant.

Next, one matches the inner and outer expansions to produce an expansion which is uniformly valid throughout the entire domain. To do so one introduces an intermediate length scale $r = \epsilon^\alpha t = \epsilon^{\alpha-1} x$ for $\alpha \in (0,1)$ and re-expresses the inner and outer expansions in terms of the new variable r,

$$\begin{aligned} u_{inner}(r) &= (1 - \epsilon^\alpha \frac{1}{r}) + \epsilon \log \epsilon \tilde{A}(1 - \frac{\epsilon^\alpha}{r}) - \epsilon \log(\epsilon^{-\alpha} r) - \epsilon^{1+\alpha} \frac{\log(\epsilon^{-\alpha} r)}{r} + \epsilon A(1 - \epsilon^\alpha \frac{1}{r}) + \dots \\ u_{outer}(r) &= 1 + \epsilon B \int_x^\infty \frac{e^{-\rho}}{\rho^2} d\rho + \dots \\ &\sim 1 + \epsilon B(\frac{\epsilon^{\alpha-1}}{r} + \log(\epsilon^{1-\alpha} r) + (\gamma - 1) + \dots). \end{aligned}$$

Matching yields conditions on the constants A, $\tilde{A}$ and B which make the two expansions equal in the overlap domain. The expansions at $\mathcal{O}(\epsilon^\alpha)$ require that $B = -1$. At $\mathcal{O}(\epsilon \log \epsilon)$, we have $\tilde{A} + \alpha = B(1 - \alpha)$ and therefore $\tilde{A} = -1$ as well. Finally, at $\mathcal{O}(\epsilon)$ we have $A = \gamma - 1$. With these constants in hand, matching can be carried out successfully, and the following uniformly valid composite expansion is obtained:

$$u(t,\epsilon) = 1 - \epsilon \int_{\epsilon t}^\infty \frac{e^{-\rho}}{\rho^2} d\rho + \frac{\epsilon \log(\epsilon)}{t} - \epsilon \frac{1-\gamma}{t} - \epsilon \frac{\log(t)}{t} + o(\epsilon). \tag{3.4}$$

We remark that the $\epsilon \log \epsilon \tilde{u}$ term that was supposed in the inner expansion is known as the logarithmic switchback term. This term is a necessary ingredient for the matched asymptotic result. If one were to instead suppose a naive perturbation expansion in powers of ϵ, then the matching would fail. Conventionally, this was a stopping point and one would have to return to the naive expansion and only after incorporating a term as above will the matching work. The power of the RG method, as we will see in the following sections, is that such a procedure is not necessary and an asymptotic expansion for the solution can be constructed using only the naive expansion.

3.2 Application of the RG method to (3.3)

We show that a uniformly valid asymptotic expansion to the solution of the Lagerstrom equation can be found by applying the RG method to the inner equation, (3.3). In the process of finding this expansion the gauge function $\epsilon\log(\epsilon)$ is naturally introduced.

3.2.1 RG_1: deriving the naive expansion

To begin, we rewrite the equation (3.3) as a system,

$$\begin{aligned} u' &= v \\ v' &= -\frac{n-1}{t}v - \epsilon uv, \end{aligned} \tag{3.5}$$

and suppose a naive perturbation expansion of the form

$$\begin{aligned} u(t,\epsilon) &= u_0(t) + \epsilon u_1(t) + \dots \\ v(t,\epsilon) &= v_0(t) + \epsilon v_1(t) + \dots, \end{aligned}$$

with arbitrary initial conditions given by $u(t_0) = w_0$ and $v(t_0) = m_0$ for some arbitrary initial time t_0. To first order, we have

$$u_0' = v_0, \quad v_0' = \frac{-2}{t}v_0.$$

The second equation is separable and can be integrated directly to find

$$v_0(t) = \frac{m_0 t_0^2}{t^2}.$$

From this, we derive u_0 by integration,

$$u_0(t) = w_0 + \int_{t_0}^{t} \frac{m_0 t_0^2}{s^2} ds = w_0 + m_0 t_0 (1 - \frac{t_0}{t}).$$

Proceeding to second order, we have

$$\begin{aligned} u_1' &= v_1 \\ v_1' &= -\frac{2}{t}v_1 - u_0 v_0. \end{aligned}$$

The initial conditions at $\mathcal{O}(\epsilon)$ are zero, because otherwise they could be incorporated into the leading order initial conditions w_0 and m_0. The equation for v_1 can be solved via the integrating factor t^2 to yield

$$v_1(t) = -m_0 t_0^2 (w_0 + m_0 t_0)\frac{t - t_0}{t^2} + \frac{m_0^2 t_0^4}{t^2}\log(\frac{t}{t_0}).$$

Next, we integrate once more to compute u_1,

$$\begin{aligned} u_1(t) &= \int_{t_0}^{t} v_1(s) ds \\ &= -m_0 t_0^2 (w_0 + m_0 t_0)\log(\frac{t}{t_0}) + m_0 t_0^2 (w_0 + m_0 t_0)(1 - \frac{t_0}{t}) \\ &- \frac{m_0^2 t_0^4}{t}\log(\frac{t}{t_0}) + m_0^2 t_0^3 (1 - \frac{t_0}{t}). \end{aligned}$$

Collecting the above results, we find that the full naive expansion of (3.3) up to and including $\mathcal{O}(\epsilon)$ is

$$\begin{aligned} u(t,\epsilon) &= w_0 + m_0 t_0 (1 - \frac{t_0}{t}) + \epsilon\left(-m_0 t_0^2 (w_0 + m_0 t_0)\log(\frac{t}{t_0}) + m_0 t_0^2 (w_0 + m_0 t_0)(1 - \frac{t_0}{t})\right. \\ &- \left.\frac{m_0^2 t_0^4}{t}\log(\frac{t}{t_0}) + m_0^2 t_0^3 (1 - \frac{t_0}{t})\right). \\ v(t,\epsilon) &= \frac{m_0 t_0^2}{t^2} + \epsilon\left(-m_0 t_0^2 (w_0 + m_0 t_0)\frac{t - t_0}{t^2} + \frac{m_0^2 t_0^4}{t^2}\log(\frac{t}{t_0})\right). \end{aligned} \tag{3.6}$$

This naive expansion is the starting point of the RG method, recall step RG_1 in (1.2).

3.2.2 RG_2: renormalizing the naive expansion

We are now ready to carry out the second step of the RG procedure, in which the naive expansion (3.6) is renormalized. To begin, we make the following choice for the first order renormalization

$$\begin{aligned} w_0 &= W - \frac{M}{t_0} + \epsilon a_1(W, M, t_0) \\ m_0 &= \frac{M}{t_0^2} + \epsilon b_1(W, M, t_0). \end{aligned} \tag{3.7}$$

This choice of renormalization is motivated by the observation that the leading order solution is described by two terms: a constant and a term that decays like t. Also, this choice of coordinate change is near-identity in the $W - M$ system. Substituting this coordinate change into the naive expansion, we find

$$\begin{aligned} u(t,\epsilon) = W - \frac{M}{t} &+ \epsilon\left(a_1 + b_1 t_0(1 - \frac{t_0}{t}) - (MW + \frac{M^2}{t})\log(\frac{t}{t_0})\right. \\ &+ \left.(MW + \frac{M^2}{t_0})(1 - \frac{t_0}{t})\right) + \mathcal{O}(\epsilon^2). \end{aligned} \tag{3.8}$$

Our goal is to select a_1 and b_1 in such a way to simplify the renormalized naive expansion by removing all non-resonant terms in (3.8). Therefore, we choose

$$\begin{aligned} a_1 &= -\frac{M^2\log(t_0)}{t_0} \\ b_1 &= -\frac{M^2}{t_0^2} + \frac{M^2\log(t_0)}{t_0^2}, \end{aligned} \tag{3.9}$$

and the renormalized expansion up to and including $\mathcal{O}(\epsilon)$ is

$$u(t,\epsilon) = W - \frac{M}{t} - \epsilon\left(MW\log(\frac{t}{t_0}) + \frac{M^2}{t}\log(t) - MW(1 - \frac{t_0}{t})\right) + \mathcal{O}(\epsilon^2). \tag{3.10}$$

All that remains in the renormalized expansion are a logarithmically divergent term, a term that does not depend on t_0 (and hence cannot be absorbed) and a term that grows linearly in t_0, namely $MW(1-(t_0/t))$, which is secular.

Remark 1. *Formally, one could remove the secular term $MW(1 - (t_0/t))$ from (3.10) by including the additional term $-\frac{MW}{t_0}$ in b_1 in the coordinate change (3.9). However, on time scales of $\mathcal{O}(1/\epsilon)$, that new term would be of the same order as the leading order term in (3.7) and, hence, would disorder that expansion.*

3.2.3 RG_3: Application of the RG condition and derivation of the final expansion

Returning to the problem at hand, we apply the RG condition to (3.10) to find

$$\begin{aligned} \frac{dW}{dt_0} &= -\epsilon\frac{MW}{t_0} + \mathcal{O}(\epsilon^2) \\ \frac{dM}{dt_0} &= -\epsilon MW + \mathcal{O}(\epsilon^2). \end{aligned} \tag{3.11}$$

These equations represent the simplest form of the differential equations for model (3.3) in that all of the non-resonant terms have been removed to this order. Moreover, the commuting diagram (1.2) suggests that the RG equations are the normal form of (3.3), up to and including $\mathcal{O}(\epsilon)$, and we will prove this in Theorem 1 below.

The final step in the RG method is to solve the RG equations (3.11) on the relevant time scale, neglecting terms of $\mathcal{O}(\epsilon^2)$ or higher. This solution is then used in (3.10) to derive a uniformly valid asymptotic expansion for the original problem.

Since we are unaware of a method to solve (3.11) directly, we use Picard iteration to derive an asymptotic approximation of the solution. The derivation is presented in Appendix A.1. The result is

$$\begin{aligned} W(t_0) &\approx W^0(t_0) := C_W - C_M + C_M \frac{e^{-\epsilon C_W (t_0-1)}}{t_0} + \epsilon C_M C_W e^{\epsilon C_W} \left(e_2(\epsilon C_W) - e_2(\epsilon C_W t_0)\right) \\ M(t_0) &\approx M^0(t_0) := C_M e^{-\epsilon C_W (t_0 - 1)}. \end{aligned}$$

We substitute these expressions into (3.10) and set $t_0 = t$, exploiting the fact that the RG condition ensures that our solution is invariant under changes in t_0 and we may then set $t_0 = t$. Note that this eliminates both terms involving MW. We then have the following expansion of the original solution:

$$\begin{aligned} u(t, \epsilon) &= W^0(t) - \frac{M^0(t)}{t} - \epsilon \frac{(M^0(t))^2 \log(t)}{t} + o(\epsilon) \\ &= C_W - C_M + \epsilon C_M C_W e^{\epsilon C_W} \left(e_2(\epsilon C_W) - e_2(\epsilon C_W t)\right) \\ &- \epsilon \frac{C_M^2 e^{-2\epsilon C_W (t-1)} \log(t)}{t} + o(\epsilon). \end{aligned} \tag{3.12}$$

We now require that the initial conditions are satisfied, namely that $u(t = 1) = 0$ and $u(t = \infty) = 1$. The first condition requires that $C_W = C_M$. The second requires that

$$1 = \epsilon C_W^2 e^{\epsilon C_W} e_2(\epsilon C_W),$$

where we have used the fact that $C_M = C_W$. We recall that as $\epsilon \to 0$, $e_2(\epsilon C_W) \sim \frac{1}{\epsilon C_W} + \log(\epsilon C_W) + (\gamma - 1) \ldots$. Therefore, we may choose $C_W = 1 - \epsilon \log(\epsilon) + \ldots$. Thus, our approximation for the solution to Lagerstrom's model equation is

$$u(t, \epsilon) = 1 - \frac{e_2(\epsilon t)}{e_2(\epsilon)} - \epsilon \frac{\log(t)}{t} + o(\epsilon). \tag{3.13}$$

Since ϵ is small, we may expand $e_2(\epsilon)$ and observe that this expansion is equivalent to the uniformly valid composite expansion in (3.4), derived by the method of matched asymptotics. Therefore, the RG method has generated the asymptotic expansion, without any foreknowledge about the logarithmic term and without any need for matching, as was first shown in [2, 3]. Also, the RG method naturally identifies the resonant term $e_2(\epsilon C_W t_0)$ which is responsible for the logarithmic switchback term.

3.3 Normal Form Analysis

We now turn our attention to the question of how accurate the asymptotic expansion, (3.13), is for small ϵ. Both the matched asymptotics and the RG method are formal procedures. There are two known proofs of accuracy for Lagerstrom's model equation. The first uses geometric desingularization, see [31, 32]. A more recent proof establishes the existence of a uniformly convergent series solution for each $n \geq 2$, [11, 12].

In this section, we show that this validity can also be established using the RG method. At this point, the extra work that we have done in computing the sub-resonant terms explicitly pays off. We appeal to the relationship between the RG method and asymptotic normal form theory to establish the validity of the asymptotic expansion (3.13).

The change of coordinates derived by the RG method, (3.7), with a_1 and b_1 given in (3.9), naturally induces a change of variables in the space of dependent variables. We will implement this coordinate transformation and show that the resulting equations are the normal form for the Lagerstrom model. By design, the normal form equations agree up to and including $\mathcal{O}(\epsilon)$ terms, and we prove a theorem (see Theorem 1) that establishes that the solutions of the original Lagerstrom model and those of the NF equations stay close to this order over long times. In this manner, we establish that (3.13) is a uniformly valid asymptotic expansion of the solution to the Lagerstrom model and, hence, justify the validity of the RG method on it.

We define a change of coordinates $\Phi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ as

$$\begin{pmatrix} u \\ v \end{pmatrix} = \Phi(U, V, t) = \begin{pmatrix} U - \frac{V}{t} - \epsilon \frac{V^2 \log(t)}{t} \\ \frac{V}{t^2} - \epsilon \frac{V^2}{t^2}(1 - \log(t)) \end{pmatrix}, \tag{3.14}$$

where $v = u'$ as above. The original problem, (3.3), can then be re-written in these new variables by

$$\begin{pmatrix} U' \\ V' \end{pmatrix} = (D\Phi)^{-1} F(\Phi) - (D\Phi)^{-1} \frac{\partial \Phi}{\partial t},$$

whenever $D\Phi^{-1}$ is invertible. To this end, we compute

$$\begin{aligned} D\Phi &= \begin{pmatrix} 1 & \frac{-1}{t} - 2\epsilon V \frac{\log(t)}{t} \\ 0 & \frac{1}{t^2} - \frac{2\epsilon V}{t^2}(1-\log(t)) \end{pmatrix} \\ F(\Phi) &= \frac{1}{t^2} \begin{pmatrix} V - \epsilon V^2(1-\log(t)) \\ \frac{-2V}{t} + 2\epsilon \frac{V^2}{t}(1-\log(t)) - \epsilon(U - \frac{V}{t} - \epsilon \frac{V^2}{t}\log(t))(V - \epsilon V^2(1-\log(t))) \end{pmatrix} \\ \frac{\partial \Phi}{\partial t} &= \frac{1}{t^2} \begin{pmatrix} V - \epsilon V^2(1-\log(t)) \\ -\frac{2V}{t} + 2\epsilon \frac{V^2}{t}(1-\log(t)) + \epsilon \frac{V^2}{t} \end{pmatrix} \end{aligned} \tag{3.15}$$

Inverting $D\Phi$ in (3.15), we find

$$D\Phi^{-1} = \frac{1}{1 - 2\epsilon V(1-\log(t))} \begin{pmatrix} 1 - 2\epsilon V(1-\log(t)) & t + 2\epsilon V t \log(t) \\ 0 & t^2 \end{pmatrix},$$

and hence $D\Phi^{-1}$ exists only whenever $1 - 2\epsilon V(1-\log(t)) \neq 0$.

Multiplying above, we arrive at the following differential equations for U and V,

$$\begin{aligned} U' &= -\epsilon \frac{UV}{t} + \frac{\epsilon^2 V^2}{t^2} \frac{-Ut(1+\log(t)) - V + 2V\log(t) - 3\epsilon V^2 \log(t) + 5\epsilon V \log(t)^2}{1 - 2\epsilon V(1-\log(t))} \\ &+ \frac{\epsilon^2 V^2}{t^2} \frac{2\epsilon U V t \log(t)(1-\log(t)) - 2\epsilon^2 V^3 \log(t)^2 (1-\log(t))}{1 - 2\epsilon V(1-\log(t))} \\ V' &= -\epsilon U V + \epsilon^2 \frac{UV^2(1-\log(t)) - V^3(1-2\log(t))/t - \epsilon V^4(1-\log(t))/t}{1 - 2\epsilon V(1-\log(t))}. \end{aligned} \tag{3.16}$$

Up to and including terms of $\mathcal{O}(\epsilon)$, the normal form (3.16) agrees with the RG equations (3.11). Hence, just as we have seen here and in [7], the RG method systematically uncovers the coordinate changes needed for the asymptotic normal form theory, recall steps RG_2 and NF in (1.2).

We will now show that the approximate solution derived via the RG method is accurate. This amounts to justifying the truncation of the $\mathcal{O}(\epsilon^2)$ terms in the normal form expression (3.16). We establish the following theorem.

Theorem 1. *Let $W^0(t)$ and $M^0(t)$ be given as above. Let $U(t)$ and $V(t)$ be the exact solutions to the full equation (3.16) with $W^0(1) = U(1)$ and $M^0(1) = V(1)$. Then, there exists $\epsilon_1 > 0$ such that for all $\epsilon \in [0, \epsilon_1)$, we have that*

$$\begin{aligned} \sup_{t\in[1,\infty)} |W^0(t) - U(t)| &= \mathcal{O}(\epsilon^2 \log^2(\epsilon)) \\ \sup_{t\in[1,\infty)} |W^0(t) - U(t)| &= \mathcal{O}(\epsilon^2 \log^2(\epsilon)). \end{aligned}$$

Moreover, under the change of coordinates (3.14) the approximate solution $\Phi(W^0, M^0, t)$ is $\mathcal{O}(\epsilon^2 \log^2(\epsilon))$ close to the true solution of the original problem.

Proof: See Appendix A.3.

■

3.4 A brief comparison to the CGO approach

Chen, Goldenfeld, and Oono analyzed the Lagerstrom model in their original work on the RG method, [2, 3]. In this section, we describe the approach taken there and how it differs from what the one detailed in the preceding sections.

The primary difference between the two approaches is that the outer formulation (3.1) of the problem is used in [2, 3]. In particular, the analysis in [2, 3] begins with the following leading order solution:

$$u(x) = A_0.$$

Clearly, no choice of this constant will make this approximation uniformly valid. Proceeding, one supposes a naive perturbation expansion in powers of an as yet to be determined gauge function $0 < \lambda(\epsilon) \ll 1$. At $\mathcal{O}(\lambda(\epsilon))$, one arrives at the following differential equation

$$\frac{d^2u_1}{dx^2} + \frac{2}{x}\frac{du_1}{dx} + A_0\frac{du_1}{dx} = 0,$$

for which a solution exists of the form,

$$u_1(x) = A_0A_1\left(e_2(A_0x_0) - e_2(A_0x)\right),$$

where A_1 is an integration constant. Given the asymptotics of the exponential integral e_2, we note that as x or x_0 vanishes, $u_1(x)$ blows up. Thus, applying the RG condition we find the RG equation is

$$\frac{dA_0}{dx_0} = \lambda(\epsilon)A_1\frac{e^{-A_0x_0}}{x_0^2} + \mathcal{O}(\lambda^2(\epsilon)).$$

This equation cannot be solved directly, but by Picard Iteration one finds a solution of the form

$$A_0(x_0) = C_A - \lambda(\epsilon)A_1\int_{x_0}^{\infty}\frac{e^{-C_A\rho}}{\rho^2}d\rho \tag{3.17}$$

where we have two unknowns, C_A the value of the function at ∞ and $\lambda(\epsilon)A_1$, the gauge function multiplied by the integration constant. Imposing the condition that $A_0(\infty) = 1$ implies that $C_A = 1$, while imposing $A_0(\epsilon) = 0$ implies that

$$\lambda(\epsilon) = \frac{1}{A_1e_2(\epsilon)}.$$

In this way, relabeling A_0 as u and x_0 as x they recover the following expansion, see (4.3) in [3],

$$u(x,\epsilon) = 1 - \frac{e_2(x)}{e_2(\epsilon)} + o(\epsilon). \tag{3.18}$$

By substituting $t = x/\epsilon$ into (3.13), we see that it's equivalent to (3.18). As noted in [3] one must go to $\mathcal{O}(\lambda(\epsilon)^{-2})$ to obtain the $-\epsilon\frac{\log(t)}{t}$ term when working with the outer equation. While the approach taken in [3] is more direct, the advantage of the approach taken here is that we are able to leverage the relationship of the RG method to asymptotic normal form in order to prove that the method is rigorous.

Remark 2. *For the Lagerstrom model problem, the outer expansion is a uniformly valid expansion on* $[1,\infty)$*, see [16, 40].*

4 Logarithmic switchback at higher order: the example of Bender and Orszag

In this section, we analyze the non-autonomous linear problem, (1.5), that was studied using matched asymptotics in [1] and treated as one of the examples in Chen, Goldenfeld and Oono [3]. As was the case with the previous examples, the asymptotic expansion of the solution of this problem leads to logarithmic gauge functions. However, these switchback terms occur at higher order here. To further contrast with the previous example, it is not known whether this logarithmic behavior can be explained using geometric desingularization. We will show that the RG method is well-equipped to compute the uniformly valid asymptotic expansion.

We recall equation (1.5),

$$\epsilon Y'' + XY' - XY = 0.$$

With the new independent variable $X = \epsilon^{1/2}t$, this system is transformed into

$$y'' + ty' - \epsilon^{1/2}ty = 0, \tag{4.1}$$

which is labeled as the inner equation. We study this equation as an initial value problem with initial conditions, $y(t_0) = A_0$ and $y'(t_0) = B_0$, for arbitrary A_0 and B_0.

The naive expansion in powers of $\sqrt{\epsilon}$ is derived up to and including terms of $\mathcal{O}(\epsilon)$ in Section 4.1, which completes the first step RG_1 of the RG method. Then, it turns out to be simplest to carry out steps RG_2 and RG_3, *i.e.* to renormalize the expansion and apply the RG condition order by order. We do so for $\mathcal{O}(\sqrt{\epsilon})$ in Section 4.2 and for $\mathcal{O}(\epsilon)$ in Section 4.3. The final RG equations are (4.12) and the expansion derived by the RG method is (4.14), see Section 4.3 below. Finally, in Section 4.4, we analyze in detail the normal form for (4.1) and the connection between the RG equations and the normal form equations. The near-identity coordinate changes, NF and RG_2, illustrated in the commuting diagram (1.2) play central roles, just as was the case in the previous two examples.

4.1 RG_1: the naive perturbation expansion

We postulate a naive perturbation expansion (in powers of $\epsilon^{1/2}$) of the solution of (4.1),

$$y(t,\epsilon) = y_0(t) + \sqrt{\epsilon}y_1(t) + \epsilon y_2(t)$$

and solve order by order. To leading order, we find

$$y_0' = x_0, \quad x_0' = -tx_0.$$

Solving, we find

$$x_0(t) = B_0 e^{t_0^2/2} e^{-t^2/2}, \quad y_0(t) = A_0 + B_0 e^{t_0^2/2}\int_{t_0}^{t} e^{-s^2/2}ds.$$

At $\mathcal{O}(\epsilon^{1/2})$, the differential equation is

$$y_1' = x_1, \quad x_1' = -tx_1 + ty_0,$$

with zero initial conditions. Solving for x_1, we find

$$x_1(t) = A_0 - A_0 e^{t_0^2/2} e^{-t^2/2} + B_0 e^{t_0^2/2}\int_{t_0}^{t} e^{-s^2/2}ds - B_0 e^{t_0^2/2}e^{-t^2/2}(t-t_0). \tag{4.2}$$

To obtain $y_1(t)$, we integrate $x_1(t)$ with zero initial condition and simplify, see Appendix B.1. Therefore, the asymptotic expansion up to and including terms of $\mathcal{O}(\epsilon^{1/2})$ is

$$\begin{aligned} y(t,\epsilon) &= A_0 + B_0 e^{t_0^2/2}\int_{t_0}^{t} e^{-s^2/2}ds + \sqrt{\epsilon}\Bigg(A_0(t-t_0) - A_0 e^{t_0^2/2}\int_{t_0}^{t} e^{-s^2/2}ds \\ &+ B_0 e^{t_0^2/2}(t+t_0)\int_{t_0}^{t} e^{-s^2/2}ds + 2B_0 e^{t_0^2/2}e^{-t^2/2} - 2B_0\Bigg). \end{aligned} \tag{4.3}$$

We now proceed to $\mathcal{O}(\epsilon)$. The initial conditions are again zero.

We solve first for x_2,

$$x_2 = e^{-t^2/2}\int_{t_0}^{t} s e^{s^2/2} y_1(s)ds. \tag{4.4}$$

The above integral is involved. We have relegated the details to Appendix B.2. We find

$$
\begin{aligned}
x_2 &= A_0 t - A_0 t_0 e^{t_0^2/2} e^{-t^2/2} - A_0 e^{-t^2/2} \int_{t_0}^{t} e^{s^2/2} ds - A_0 t_0 + A_0 t_0 e^{t_0^2/2} e^{-t^2/2} \\
&- A_0 e^{t_0^2/2} \int_{t_0}^{t} e^{-s^2/2} ds + A_0 e^{t_0^2/2} e^{-t^2/2} (t - t_0) + B_0 e^{t_0^2/2} t \int_{t_0}^{t} e^{-s^2/2} ds \\
&- B_0 e^{t_0^2/2} e^{-t^2/2} \int_{t_0}^{t} e^{s^2/2} \int_{t_0}^{s} e^{-\sigma^2/2} d\sigma ds - B_0 e^{t_0^2/2} e^{-t^2/2} \left(\frac{t^2}{2} - \frac{t_0^2}{2} \right) \\
&+ B_0 t_0 e^{t_0^2/2} \int_{t_0}^{t} e^{-s^2/2} ds - B_0 t_0 e^{t_0^2/2} e^{-t^2/2} (t - t_0) + B_0 e^{t_0^2/2} e^{-t^2/2} (t^2 - t_0^2) \\
&- 2B_0 + 2B_0 e^{-t^2/2} e^{t_0^2/2}. \qquad (4.5)
\end{aligned}
$$

As was the case at the previous order, to find $y_2(t)$ we integrate $x_2(t)$ while imposing zero initial conditions. In Appendix B.2, we calculate

$$
\begin{aligned}
y_2(t) &= A_0 \frac{(t - t_0)^2}{2} - A_0 \int_{t_0}^{t} e^{-s^2/2} \int_{t_0}^{s} e^{\sigma^2/2} d\sigma ds - A_0 e^{t_0^2/2} (t + t_0) \int_{t_0}^{t} e^{-s^2/2} ds \\
&- 2A_0 e^{t_0^2/2} e^{-t^2/2} + 2A_0 + B_0 e^{t_0^2/2} \frac{(t + t_0)^2}{2} \int_{t_0}^{t} e^{-s^2/2} ds \\
&- B_0 e^{t_0^2/2} \int_{t_0}^{t} e^{-s^2/2} \int_{t_0}^{s} e^{\sigma^2/2} \int_{t_0}^{\sigma} e^{-\tau^2/2} d\tau d\sigma ds \\
&+ 2B_0 t_0 e^{t_0^2/2} e^{-t^2/2} + 2B_0 e^{t_0^2/2} \int_{t_0}^{t} e^{-s^2/2} ds - 2B_0 t. \qquad (4.6)
\end{aligned}
$$

This completes step RG_1, the derivation of the naive expansion up to and including $\mathcal{O}(\epsilon)$.

4.2 The renormalized expansion and RG equations up to and including $\mathcal{O}(\sqrt{\epsilon})$

At this point, it is more convenient to perform the steps RG_2 and RG_3 order by order. In this section, we present the analysis up to and including terms of $\mathcal{O}(\sqrt{\epsilon})$.

We renormalize the initial conditions as follows,

$$
\begin{aligned}
A_0 &= \sum_{k=0}^{\infty} a_k(A, B, t_0) \epsilon^{k/2} \\
B_0 &= \sum_{k=0}^{\infty} b_k(A, B, t_0) \epsilon^{k/2},
\end{aligned}
$$

selecting a_k and b_k so as to remove non-resonant terms from the naive expansion. We begin at $\mathcal{O}(1)$, where we choose a_0 and b_0 so as to remove all explicit reference to the initial time t_0. Hence,

$$
\begin{aligned}
a_0 &= A - B \int_{t_0}^{0} e^{-s^2/2} ds \\
b_0 &= B e^{-t_0^2/2}.
\end{aligned}
$$

Then the renormalized expansion up to and including $\mathcal{O}(\sqrt{\epsilon})$ is

$$
\begin{aligned}
y(t, \epsilon) &= A + B \int_{0}^{t} e^{-s^2/2} ds \\
&+ \sqrt{\epsilon} \int_{t_0}^{t} e^{-s^2/2} ds \left(b_1 e^{t_0^2/2} - A e^{t_0^2/2} + B e^{t_0^2/2} \int_{t_0}^{0} e^{-s^2/2} ds + B(t + t_0) \right) \\
&+ \sqrt{\epsilon} \left(a_1 + A(t - t_0) - B(t - t_0) \int_{t_0}^{0} e^{-s^2/2} ds + 2B e^{-t^2/2} - 2B e^{-t_0^2/2} \right).
\end{aligned}
$$

Splitting the first integral at $\mathcal{O}(\sqrt{\epsilon})$ and grouping like terms, we find

$$\begin{aligned} y(t,\epsilon) &= A + B\int_0^t e^{-s^2/2}ds \\ &+ \sqrt{\epsilon}\int_0^t e^{-s^2/2}ds\left(b_1 e^{t_0^2/2} - Ae^{t_0^2/2} + Be^{t_0^2/2}\int_{t_0}^0 e^{-s^2/2}ds + B(t+t_0)\right) \\ &+ \sqrt{\epsilon}\int_{t_0}^0 e^{-s^2/2}ds\left(b_1 e^{t_0^2/2} - Ae^{t_0^2/2} + Be^{t_0^2/2}\int_{t_0}^0 e^{-s^2/2}ds + B(t+t_0) - B(t-t_0)\right) \\ &+ \sqrt{\epsilon}\left(a_1 + A(t-t_0) + 2Be^{-t^2/2} - 2Be^{-t_0^2/2}\right). \end{aligned}$$

Analysis of the final two lines above suggests making the following choices for the renormalization at first order so that all non-resonant terms are removed from the expansion at $\mathcal{O}(\sqrt{\epsilon})$:

$$\begin{aligned} a_1 &= 2Be^{-t_0^2/2} \\ b_1 &= A - B\int_{t_0}^0 e^{-s^2/2}ds - 2Bt_0 e^{-t_0^2/2}. \end{aligned} \tag{4.7}$$

Thus, the renormalized expansion up to and including $\mathcal{O}(\sqrt{\epsilon})$ is

$$y(t,\epsilon) = A + B\int_0^t e^{-s^2/2}ds + \sqrt{\epsilon}B(t-t_0)\int_0^t e^{-s^2/2}ds + \sqrt{\epsilon}\left(A(t-t_0) + 2Be^{-t^2/2}\right). \tag{4.8}$$

We observe that the non-resonant and sub-resonant terms dependent on the initial time t_0 have been removed from the expansion. The only t_0 dependent terms remaining are secular terms.

We now apply the RG condition. Noting that both A' and B' must be $\mathcal{O}(\sqrt{\epsilon})$, we find that up to and including terms of $\mathcal{O}(\sqrt{\epsilon})$

$$\frac{\partial y}{\partial t_0} = A' + B'\int_0^t e^{-s^2/2}ds + \sqrt{\epsilon}\left(-A - B\int_0^t e^{-s^2/2}ds\right) = 0.$$

This generates the first order RG equations

$$\begin{aligned} A' &= \sqrt{\epsilon}A \\ B' &= \sqrt{\epsilon}B. \end{aligned} \tag{4.9}$$

4.3 Renormalized expansion and the RG equations up to and including terms of $\mathcal{O}(\epsilon)$

To carry out steps RG_2 and RG_3 to $\mathcal{O}(\epsilon)$, we must first analyze the asymptotics of the integrals in (4.6) for $y_2(t)$. It can be shown that for fixed t_0, both integrals diverge as $t \to \infty$. Using l'Hopital's rule we can derive the following rates of divergence (note the different limits of integration),

$$\begin{aligned} \int_0^t e^{-s^2/2}\int_0^s e^{\sigma^2/2}d\sigma ds &= \left(\frac{2}{\sqrt{\pi}}\log(t) + \eta(t)\right)\int_0^t e^{-s^2/2}ds \\ \int_0^t e^{-s^2/2}\int_0^s e^{\sigma^2/2}\int_0^\sigma e^{-\tau^2/2}d\tau d\sigma ds &= (\log(t) + \nu(t))\int_0^t e^{-s^2/2}ds, \end{aligned}$$

where $\eta(\cdot)$ and $\nu(\cdot)$ are bounded or diverge slower that $\log(\cdot)$.

Remark 3. *This logarithmic divergence can also be observed using integration by parts. For example,*

$$\begin{aligned} \int_{t_0}^t e^{-s^2/2}\int_{t_0}^s e^{\sigma^2/2}d\sigma ds &= \int_{t_0}^t e^{-s^2/2}\left(\frac{e^{s^2/2}}{s} - \frac{e^{t_0^2/2}}{t_0} + \int_{t_0}^s \frac{e^{\sigma^2/2}}{\sigma^2}d\sigma\right)ds \\ &= \log(\frac{t}{t_0}) - \frac{e^{t_0^2}}{t_0}\int_{t_0}^t e^{-s^2/2}ds + \int_{t_0}^t e^{-s^2/2}\int_{t_0}^s \frac{e^{\sigma^2/2}}{\sigma^2}d\sigma ds. \end{aligned}$$

The final integral is clearly bounded and therefore the logarithmic asymptotics are justified.

Thus, we may rewrite the second A_0 term in $y_2(t)$ as

$$\begin{aligned} A_0 \int_{t_0}^{t} e^{-s^2/2} \int_{t_0}^{s} e^{\sigma^2/2} d\sigma ds &= \int_{0}^{t} e^{-s^2/2} ds \left(\frac{2A_0}{\sqrt{\pi}} \log(\frac{t}{t_0}) + A_0 \eta(t) - A_0 \eta(t_0) \right) \\ &- A_0 \int_0^{t_0} e^{s^2/2} ds \int_{t_0}^{t} e^{-s^2/2} ds, \end{aligned}$$

and the second B_0 term in $y_2(t)$ as

$$\int_{t_0}^{t} e^{-s^2/2} \int_{t_0}^{s} e^{\sigma^2/2} \int_{t_0}^{\sigma} e^{-\tau^2/2} d\tau d\sigma ds = \xi(t_0) \int_{t_0}^{t} e^{-s^2/2} ds$$

$$+ \left(\log(\frac{t}{t_0}) + \nu(t) - \nu(t_0) - \int_{t_0}^{0} e^{-s^2/2} ds \left(\frac{2}{\sqrt{\pi}} \log(\frac{t}{t_0}) + \eta(t) - \eta(t_0) \right) \right) \int_0^t e^{-s^2/2} ds,$$

where $\xi(t_0)$ diverges slower than $e^{t_0^2/2}$ and is given by

$$\xi(t_0) = \int_0^{t_0} e^{-s^2/2} ds \int_0^{t_0} e^{s^2/2} ds - \int_0^{t_0} e^{s^2/2} \int_0^{s} e^{-\sigma^2/2} d\sigma ds.$$

To simplify our calculations we first write the renormalization at first order in terms of the original initial conditions A_0 and B_0 and reduce the $\mathcal{O}(\epsilon)$ terms. We see that the change of variables at $\mathcal{O}(\sqrt{\epsilon})$ is

$$\begin{aligned} a_1 &= 2B_0 \\ b_1 &= A_0 - 2B_0 t_0. \end{aligned}$$

Therefore, the terms at $\mathcal{O}(\epsilon)$ are:

$$\begin{aligned} y_2(t) &= a_2 + b_2 e^{t_0^2/2} \int_{t_0}^{t} e^{-s^2/2} ds + 2B_0 t_0 - 2B_0 e^{t_0^2/2} t_0^2 \int_{t_0}^{t} e^{-s^2/2} ds + A_0 \frac{(t-t_0)^2}{2} \\ &+ 2B_0 t_0 e^{t_0^2/2} e^{-t^2/2} + B_0 e^{t_0^2/2} \frac{(t-t_0)^2}{2} \int_{t_0}^{t} e^{-s^2/2} ds \\ &- \frac{2}{\sqrt{\pi}} \log(\frac{t}{t_0}) \left(A_0 - B_0 e^{t_0^2/2} \int_{t_0}^{0} e^{-s^2/2} ds \right) \int_0^t e^{-s^2/2} ds \\ &- \left(A_0 - B_0 e^{t_0^2/2} \int_{t_0}^{0} e^{-s^2/2} ds \right) (\eta(t) - \eta(t_0)) \int_0^t e^{-s^2/2} ds \\ &- B_0 e^{t_0^2/2} \log(\frac{t}{t_0}) \int_0^t e^{-s^2/2} ds - B_0 e^{t_0^2/2} (\nu(t) - \nu(t_0)) \int_0^t e^{-s^2/2} ds \\ &+ A_0 \int_0^{t_0} e^{s^2/2} ds \int_{t_0}^{t} e^{-s^2/2} ds - B_0 e^{t_0^2/2} \xi(t_0) \int_{t_0}^{t} e^{-s^2/2} ds. \end{aligned}$$

We now apply our leading order change of variables:

$$
\begin{aligned}
y_2(t) &= a_2 + b_2 e^{t_0^2/2}\int_{t_0}^{t} e^{-s^2/2}ds + 2Bt_0e^{-t_0^2/2} - 2Bt_0^2\int_{t_0}^{t} e^{-s^2/2}ds + A\frac{(t-t_0)^2}{2} \\
&- 2Bt_0e^{-t^2/2} - B\frac{(t-t_0)^2}{2}\int_{t_0}^{0} e^{-s^2/2}ds + B\frac{(t-t_0)^2}{2}\int_{t_0}^{t} e^{-s^2/2}ds \\
&- A\frac{2}{\sqrt{\pi}}\log(\frac{t}{t_0})\int_{0}^{t} e^{-s^2/2}ds - A\left(\eta(t)-\eta(t_0)\right)\int_{0}^{t} e^{-s^2/2}ds \\
&- B\log(\frac{t}{t_0})\int_{0}^{t} e^{-s^2/2}ds - B\left(\nu(t)-\nu(t_0)\right)\int_{0}^{t} e^{-s^2/2}ds \\
&+ A\int_{0}^{t_0} e^{s^2/2}ds\int_{t_0}^{t} e^{-s^2/2}ds - B\int_{t_0}^{0} e^{-s^2/2}ds\int_{0}^{t_0} e^{s^2/2}ds\int_{t_0}^{t} e^{-s^2/2}ds \\
&- B\xi(t_0)\int_{t_0}^{t} e^{-s^2/2}ds.
\end{aligned}
$$

There are six secular terms in this expression. Namely those that diverge with $(t-t_0)^2$ rate, those with $\log(t)$ rate, and those with rate $\nu(t)$. We make the following choice for the change of variables at $\mathcal{O}(\epsilon)$ to remove the remaining non-resonant terms:

$$
\begin{aligned}
a_2 &= -2Bt_0e^{-t_0^2/2} \\
b_2 &= 2Bt_0^2e^{-t_0^2/2} - A\eta(t_0)e^{-t_0^2/2} - B\nu(t_0)e^{-t_0^2/2} \\
&- Ae^{-t_0^2/2}\int_{0}^{t_0} e^{s^2/2}ds + Be^{-t_0^2/2}\int_{t_0}^{0} e^{-s^2/2}ds\int_{0}^{t_0} e^{s^2/2}ds + B\xi(t_0)e^{-t_0^2/2}.
\end{aligned} \tag{4.10}
$$

Thus, the terms of $\mathcal{O}(\epsilon)$ in renormalized expansion are

$$
\begin{aligned}
y_2(t) &= A\frac{(t-t_0)^2}{2} + \left(B\frac{(t-t_0)^2}{2} - A\frac{2}{\sqrt{\pi}}\log(\frac{t}{t_0}) - B\log(\frac{t}{t_0})\right)\int_{0}^{t} e^{-s^2/2}ds \\
&- \left(A\eta(t) + B\nu(t)\right)\int_{0}^{t} e^{-s^2/2}ds - 2Bt_0e^{-t^2/2}.
\end{aligned}
$$

Together with the lower order terms in the renormalized expansion in (4.8), we find that the full renormalized expansion up to and including terms of $\mathcal{O}(\epsilon)$ is

$$
\begin{aligned}
y(t,\epsilon) &= A + B\int_{0}^{t} e^{-s^2/2}ds + \sqrt{\epsilon}B(t-t_0)\int_{0}^{t} e^{-s^2/2}ds + \sqrt{\epsilon}\left(A(t-t_0) + 2Be^{-t^2/2}\right) \\
&+ \epsilon A\frac{(t-t_0)^2}{2} + \epsilon\left(B\frac{(t-t_0)^2}{2} - A\frac{2}{\sqrt{\pi}}\log(\frac{t}{t_0}) - B\log(\frac{t}{t_0})\right)\int_{0}^{t} e^{-s^2/2}ds \\
&- \epsilon\left(A\eta(t) + B\nu(t)\right)\int_{0}^{t} e^{-s^2/2}ds - \epsilon 2Bt_0e^{-t^2/2}.
\end{aligned} \tag{4.11}
$$

Finally, we apply the RG condition to the renormalized expansion (4.11) to find the RG equations

$$
\begin{aligned}
\frac{dA}{dt_0} &= \sqrt{\epsilon}A + \mathcal{O}(\epsilon^{3/2}) \\
\frac{dB}{dt_0} &= \sqrt{\epsilon}B - \epsilon\left(\frac{2A}{\sqrt{\pi}t_0} + \frac{B}{t_0}\right) + \mathcal{O}(\epsilon^{3/2}).
\end{aligned} \tag{4.12}
$$

These differential equations can be solved exactly as follows,

$$
\begin{aligned}
A(t_0) &= C_1e^{\sqrt{\epsilon}t_0} \\
B(t_0) &= C_2t_0^{-\epsilon}e^{\sqrt{\epsilon}t_0} - \epsilon\frac{2C_1}{\sqrt{\pi}}e^{\sqrt{\epsilon}t_0}.
\end{aligned}
$$

Thus, the approximate solution given by the RG method is

$$y(t,\epsilon) = C_1 e^{\sqrt{\epsilon}t} + \left(C_2 t^{-\epsilon} e^{\sqrt{\epsilon}t} - \epsilon \frac{2C_1}{\sqrt{\pi}} e^{\sqrt{\epsilon}t} \right) \int_0^t e^{-s^2/2} ds. \tag{4.13}$$

Enforcement of the left hand boundary condition requires that $C_1 = 0$, while the right hand condition requires

$$C_2 \epsilon^{\epsilon/2} \int_0^{1/\sqrt{\epsilon}} e^{-s^2/2} ds = 1.$$

This implies that the RG approximation, out to the order that we calculated, is given by

$$y(t,\epsilon) = \epsilon^{-\epsilon/2} t^{-\epsilon} e^{\sqrt{\epsilon}t} \frac{\int_0^t e^{-s^2/2} ds}{\int_0^{1/\sqrt{\epsilon}} e^{-s^2/2} ds}. \tag{4.14}$$

Here we note that the denominator is $\sqrt{\pi/2}$ plus exponentially small terms.

4.4 Normal Form theory for (4.1)

In this section, we carry out a normal form analysis of equation (4.1). First, we briefly follow the standard normal form theory and derive the normal form, (4.16), below. Then, we apply the near-identity coordinate change identified by the RG method to (4.1) and determine the normal form induced by the RG procedure, see (4.17), below. These calculations are carried out to $\mathcal{O}(\sqrt{\epsilon})$ to illustrate results. Higher order equivalents may also be derived.

We begin with the inner equation as a system,

$$\begin{aligned} x' &= -tx + \sqrt{\epsilon} t y \\ y' &= x. \end{aligned} \tag{4.15}$$

We seek a linear, near-identity change of coordinates

$$\begin{aligned} x &= w + \sqrt{\epsilon} a w + \sqrt{\epsilon} b z \\ y &= z + \sqrt{\epsilon} c w + \sqrt{\epsilon} d z. \end{aligned}$$

Plugging this into (4.15) we find

$$\begin{aligned} w' &= -tw + \sqrt{\epsilon}\left(-aw' - bz' - taw - tbz + tz\right) + \mathcal{O}(\epsilon) \\ z' &= w + \sqrt{\epsilon}\left(-cw' - dz' + aw + bz\right). \end{aligned}$$

To leading order $w' = -tw$ and $z' = w$. Hence, we find

$$\begin{aligned} w' &= -tw + \sqrt{\epsilon}\left(taw - bw - taw - tbz + tz\right) + \mathcal{O}(\epsilon) \\ z' &= w + \sqrt{\epsilon}\left(tcw - dw + aw + bz\right) + \mathcal{O}(\epsilon). \end{aligned}$$

Thus, if we choose $b = 1$ and $a = d$ we find that the first order linear normal form for the system in (4.15) is given by

$$\begin{aligned} w' &= -tw - \sqrt{\epsilon} w + \mathcal{O}(\epsilon) \\ z' &= w + \sqrt{\epsilon}\left(tcw + z\right) + \mathcal{O}(\epsilon), \end{aligned} \tag{4.16}$$

where we leave c undetermined.

We now contrast the above normal form with the one found using RG. We will perform the RG calculation in a slightly different manner than before. We begin by recalling the naive solution up to and including terms of $\mathcal{O}(\sqrt{\epsilon})$ (see (4.3)),

$$\begin{aligned} y(t,\epsilon) = A_0 \quad &+ \quad B_0 e^{t_0^2/2} \int_{t_0}^t e^{-s^2/2} ds \\ &+ \quad \epsilon^{1/2} \Big(A_0(t - t_0) - A_0 e^{t_0^2/2} \int_{t_0}^t e^{-s^2/2} ds + B_0 e^{t_0^2/2}(t + t_0) \int_{t_0}^t e^{-s^2/2} ds \\ &+ \quad 2B_0 e^{t_0^2/2} e^{-t^2/2} - 2B_0 \Big). \end{aligned}$$

We make the following change of coordinates for A_0 and B_0,

$$\begin{aligned} B_0 &= B + \sqrt{\epsilon} A \\ A_0 &= A + \sqrt{\epsilon} 2B. \end{aligned}$$

To facilitate a comparison with the Normal Form approach, we note that A_0 corresponds to y and B_0 corresponds to x, which explains the non-alphabetical ordering of these terms. Upon performing this change of variables, we find that the expansion is reduced to

$$\begin{aligned} y(t) = A &+ B e^{t_0^2/2} \int_{t_0}^{t} e^{-s^2/2} ds \\ &+ \epsilon^{1/2} \left(A(t - t_0) + B e^{t_0^2/2} (t + t_0) \int_{t_0}^{t} e^{-s^2/2} ds + 2B e^{t_0^2/2} e^{-t^2/2} \right). \end{aligned}$$

Applying the RG condition and after some reduction, we find the following RG equation

$$\begin{aligned} \frac{\partial B}{\partial t_0} &= -B t_0 - \sqrt{\epsilon} B \\ \frac{\partial A}{\partial t_0} &= B + \sqrt{\epsilon} \left(2 B t_0 + A\right). \end{aligned} \tag{4.17}$$

We therefore see that the normal form transformation prescribed by the RG procedure is of the class given by the normal form theory ($b = 1$, $a = d = 0$ and $c = 2$). However, we see a certain advantage to the RG procedure if we asymptotically solve the two systems (4.16) and (4.17). The solution to (4.16) is

$$\begin{aligned} z(t, \epsilon) &= A + B e^{t_0^2/2} \int_{t_0}^{t} e^{-s^2/2} ds + \epsilon^{1/2} \left(A(t - t_0) + B e^{t_0^2/2} (t + t_0) \int_{t_0}^{t} e^{-s^2/2} ds \right. \\ &- \left. (c - 2) B (1 - e^{t_0^2/2} e^{-t^2/2}) \right), \end{aligned}$$

while the solution for (4.17) is

$$A(t_0) = A(T_0) + B(T_0) e^{T_0^2/2} \int_{T_0}^{t_0} e^{-s^2/2} ds + \epsilon^{1/2} \left(A(T_0)(t_0 - T_0) + B(T_0) e^{T_0^2/2} (t_0 + T_0) \int_{T_0}^{t_0} e^{-s^2/2} ds \right).$$

To conclude, we observe that the NF method prescribed a one parameter family of changes of coordinates, specified by the parameter c. The RG method shows that one of these changes of coordinates, with $c = 2$, is distinguished. Working only on the space of planar vector fields, it is not clear from the Normal Form procedure why this particular coordinate change is preferential to others. It is only after computing the solution that this property is made clear.

5 Discussion

The three examples discussed in this article demonstrate that the RG method readily handles perturbation problems in which logarithmic switchback terms arise. As we have outlined throughout the text, this feature can be understood through the close relationship enjoyed between the RG method and asymptotic normal form theory. In this discussion, we briefly comment on this relationship and its pertinence to logarithmic switchback phenomena.

We consider a general perturbation problem

$$u' = A(t)u + \epsilon F(u).$$

Let $\Phi(t, t_0)$ be the fundamental matrix solution associated to the non-autonomous linear system. The naive asymptotic expansion up to and including terms of $\mathcal{O}(\epsilon)$ is

$$u(t, \epsilon) = \Phi(t, t_0) u(t_0) + \epsilon \int_{t_0}^{t} \Phi(t, s) F(\Phi(s, t_0) u(t_0)) ds + \mathcal{O}(\epsilon^2).$$

Problems arise when the $\mathcal{O}(\epsilon)$ terms become larger than those at leading order. At this juncture, we can proceed to identify resonant and sub-resonant terms in $F(u)$ according to whether the above integral remains bounded. Suppose that the integral is unbounded and, furthermore, that the nonlinearity can be divided into resonant and sub-resonant pieces, i.e. $F(u) = F^R(u) + F^{SR}(u)$. It is these resonant terms that encode the logarithmic switchback terms.

Having identified the resonant terms, the next step of the RG procedure is to isolate these terms in a suitable fashion. As we described in this article, one method to accomplish this is to renormalize the initial conditions in such a way as to remove all instances of the initial time t_0 except for those that occur in the resonant terms. That is, we introduce a new initial time T_0 and make the near-identity change of coordinates

$$U(t_0) = \Phi(T_0, t_0)u(t_0) + \epsilon \int_{t_0}^{T_0} \Phi(T_0, s)F^{SR}(\Phi(s, t_0)u(t_0))ds.$$

Here, T_0 is used only to split the integral into a time dependent term and one dependent only on t_0 and will have no bearing on the final result. Alternately, an indefinite integral could be used. The renormalized expansion then becomes

$$u(t, \epsilon) = \Phi(t, T_0)U(t_0) + \epsilon \int_{T_0}^{t} \Phi(t, s)F^{SR}(\Phi(s, T_0)U(t_0))ds + \epsilon \int_{t_0}^{t} \Phi(t, s)F^{R}(\Phi(s, T_0)U(t_0))ds + \mathcal{O}(\epsilon^2). \quad (5.1)$$

Finally, applying the RG condition yields the familiar RG (or averaged) equation

$$\frac{dU}{dt_0} = \epsilon\Phi(T_0, t_0)F^R(\Phi(t_0, T_0)U(t_0)) + \mathcal{O}(\epsilon^2). \quad (5.2)$$

If one can find a bounded solution to this differential equation, then the solution can be substituted back into the renormalized expansion (5.1) and a uniformly valid asymptotic expansion up to and including terms of $\mathcal{O}(\epsilon)$ is found after evaluating at $t_0 = t$, effectively replacing the unbounded integral in that formula with the bounded solution of the averaged equation (5.2).

This can be clearly observed in the case of the Lagerstrom model studied in section 3. Indeed, equation (5.2) is the system given in (3.11). The solution of (3.11) involves the function $\epsilon e_2(\epsilon t)$, as shown in section 3.2.3. This function is bounded, but when expanded in powers of ϵ the resulting asymptotic expansion is not well ordered for large t. In this way, the resonant terms in the renormalized expansion (3.10) are remnants of the expansion of $e_2(\epsilon t)$ in powers of ϵ. Computing the solution of (3.11) and substituting this back into the renormalized expansion, we see that the resonant terms are replaced by the appropriate bounded term that contains the logarithmic switchback present in the problem.

Before concluding, we make one final observation about the RG procedure. It bears mentioning that above method is not the only means by which to renormalize the asymptotic expansion. Indeed, one could instead isolate the resonant terms by "marking" them with the introduction of an arbitrary auxiliary variable τ. One splits the resonant integral into two integrals

$$\begin{aligned} u(t, \epsilon) &= \Phi(t, t_0)u(t_0) + \epsilon \int_{t_0}^{\tau} \Phi(t, s)F^R(\Phi(s, t_0)u(t_0))ds + \epsilon \int_{\tau}^{t} \Phi(t, s)F^R(\Phi(s, t_0)u(t_0))ds \\ &+ \epsilon \int_{t_0}^{t} \Phi(t, s)F^{SR}(\Phi(s, t_0)u(t_0))ds + \mathcal{O}(\epsilon^2) \end{aligned}$$

and then absorbs the time independent one into a new constant of integration $U(\tau)$. One then derives

$$u(t, \epsilon) = \Phi(t, t_0)U(\tau) + \epsilon \int_{\tau}^{t} \Phi(t, s)F^R(\Phi(s, t_0)u(t_0))ds + \epsilon \int_{t_0}^{t} \Phi(t, s)F^{SR}(\Phi(s, t_0)u(t_0))ds + \mathcal{O}(\epsilon^2).$$

Since τ is arbitrary, we may differentiate the above equation and derive the exact same equation as in (5.2). This is, in essence, the procedure adopted in [2, 3]. Note that if one is only interested in computing an asymptotic expansion of the solution, this is the most direct method by which to do that. Alternatively, the approach taken in this article makes the relationship of the RG method to normal form theory explicit and allows for the possibility of rigorous analysis via normal form theory techniques.

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A Justification of the Picard iteration and a proof of Theorem 1

A.1 Derivation of the approximate solution to the RG equations (3.11)

Our first observation is that equations (3.11) are bounded and Lipschitz on a neighborhood of the origin for all $t_0 \geq 1$. Therefore we expect their solutions to exist locally although we cannot compute them explicitly. That being said, we note that we can reduce this system of differential equations to a single differential equation by rescaling time. We let

$$\tau = \int_1^{t_0} W(s)ds,$$

and note that we may then solve

$$M(t_0) = C_M e^{-\epsilon \int_1^{t_0} W(s)ds}.$$

In turn, plugging this into the equation for W yields the following differential equation

$$W' = -\epsilon \frac{C_M W e^{-\epsilon \int_1^{t_0} W(s)ds}}{t_0}. \tag{A.1}$$

Again, this solution cannot be written down exactly, but we may use Picard Iteration to write down an approximate solution. Namely, we define an integral operator $T\phi$,

$$T\phi = C_W - \epsilon C_M \int_1^{t_0} \phi(s) \frac{e^{-\epsilon \int_1^s \phi(\sigma)d\sigma}}{s} ds.$$

A solution of (A.1) is given as a fixed point of this operator. We suppose that the solution is given by the constant C_W. This guess is inserted into $T\phi$. In this way we define

$$\begin{aligned} W^0 = TC_W &= C_W - \epsilon C_M C_W \int_1^{t_0} \frac{e^{-\epsilon C_W (s-1)}}{s} ds \\ &= C_W - \epsilon C_M C_W e^{\epsilon C_W} \left(e_1(\epsilon C_W) - e_1(\epsilon C_W t_0) \right). \end{aligned}$$

We recall that the exponential integrals e_1 and e_2 are related via the identity

$$x e_1(x) = e^{-x} - x e_2(x).$$

Substituting this into the equation for W^0, we find

$$W^0(t_0) = C_W - C_M + C_M \frac{e^{-\epsilon C_W (t_0 - 1)}}{t_0} + \epsilon C_M C_W e^{\epsilon C_W} \left(e_2(\epsilon C_W) - e_2(\epsilon C_W t_0) \right).$$

With this in hand we specify,

$$\begin{aligned} M^0(t_0) &:= C_M e^{-\epsilon \int_1^{t_0} W^0(s)ds} \\ &\approx C_M e^{-\epsilon C_W (t_0 - 1)}. \end{aligned}$$

We emphasize that this is an approximation. Note that W^0 differs from C_W by $\mathcal{O}(\epsilon \log(\epsilon))$ amounts. This is not an issue, however, because these differences are not significant until $\mathcal{O}(1/\epsilon^2 \log(\epsilon))$ timescales, at which point M^0 is already $o(\epsilon)$.

A.2 Justification of the Picard Iteration

In the previous section, we constructed the asymptotic approximation (3.13) to the solution of Lagerstrom's model equation, (3.1). The approximation relied heavily upon W^0 and M^0, which are the first iterates of a Picard iteration scheme. We will now show that these iterates are, in fact, valid asymptotic approximations to the solutions of the RG equation (3.11).

Suppose that C_W and C_M are given so that as $\epsilon \to 0$, both constants converge to a fixed, strictly positive real number. Then pick $0 < \beta < C_W/5$ independent of ϵ and define

$$S := \left\{ \phi \in C^0([1,\infty), [C_W - \beta, C_W + \beta]) \quad | \quad \phi(1) = C_W \right\}. \tag{A.2}$$

For any choice of β or C_W, S is a complete metric space in the supnorm topology. We also note that for all $\phi \in S$,

$$\begin{aligned} |T\phi - C_W| &\leq \epsilon C_M \int_1^t |\frac{\phi(s)e^{-\epsilon \int_1^s \phi(\sigma)d\sigma}}{s}|ds \\ &\leq \epsilon C_M (C_W + \beta) \int_1^t |\frac{e^{-\epsilon(C_W-\beta)(s-1)}}{s}|ds \\ &\leq \epsilon C_M (C_W + \beta) \left(e_1(\epsilon(C_W - \beta)) - e_1(\epsilon(C_W - \beta)t) \right). \end{aligned}$$

Therefore, we may select ϵ sufficiently small so that $||T\phi - C_W||_S \leq \beta$ and $T : S \to S$. We now must show that T is a contraction. Let ϕ and ψ be elements of the space S and consider

$$\begin{aligned} |T\phi - T\psi| &= |\epsilon C_M \int_1^t \frac{1}{s} \left(\phi(s)e^{-\epsilon \int_1^s \phi(\sigma)d\sigma} - \psi(s)e^{-\epsilon \int_1^s \psi(\sigma)d\sigma} \right) ds| \\ &= |\epsilon C_M \int_1^s \frac{e^{-\frac{\epsilon}{2}\int_1^s \phi(\sigma)d\sigma}}{s} \left(\phi(s)e^{-\frac{\epsilon}{2}\int_1^s \phi(\sigma)d\sigma} - \psi(s)e^{-\frac{\epsilon}{2}\int_1^s (2\psi(\sigma)-\phi(\sigma))\sigma}. \right) ds| \end{aligned}$$

From the above calculations we know that if we can bound the term inside the parenthesis by $||\phi - \psi||_S$ then we may choose ϵ small enough that T will be a contraction. Therefore, we set

$$\begin{aligned} F(s) &= |\phi(s)e^{-\frac{\epsilon}{2}\int_1^s \phi(\sigma)d\sigma} - \psi(s)e^{-\frac{\epsilon}{2}\int_1^s (2\psi(\sigma)-\phi(\sigma))\sigma}| \\ &\leq e^{-\frac{\epsilon}{2}\int_1^s \phi(\sigma)d\sigma}|\phi - \psi| + \psi e^{-\frac{\epsilon}{2}\int_1^s \psi(\sigma)d\sigma}|e^{\frac{\epsilon}{2}\int_1^s (\psi(\sigma)-\phi(\sigma))d\sigma} - e^{-\frac{\epsilon}{2}\int_1^s (\psi(\sigma)-\phi(\sigma))d\sigma}| \\ &= e^{-\frac{\epsilon}{2}\int_1^s \phi(\sigma)d\sigma}|\phi - \psi| + 2\psi e^{-\frac{\epsilon}{2}\int_1^s \psi(\sigma)d\sigma}|\sinh(\frac{\epsilon}{2}\int_1^s (\psi(\sigma) - \phi(\sigma))d\sigma)|. \end{aligned} \tag{A.3}$$

The first term is clearly bounded by $||\phi - \psi||_S$. We label the second term $F^{II}(s)$ and estimate as follows,

$$\begin{aligned} F^{II}(s) &= 2\psi e^{-\frac{\epsilon}{2}\int_1^s \psi(\sigma)d\sigma}|\sinh(\frac{\epsilon}{2}\int_1^s (\psi(\sigma) - \phi(\sigma))d\sigma)| \\ &\leq 2(C_W + \beta)e^{-\frac{\epsilon}{2}(C_W-\beta)(s-1)} \sinh(\frac{\epsilon}{2}||\phi - \psi||_S(s-1)). \end{aligned}$$

We note that $F^{II}(1) = 0$ and that since $||\phi - \psi||_S < (C_W - \beta)$ then $F^{II}(\infty) = 0$ as well. All that remains to show is that F^{II} remains well behaved on the interior. To do that we compute that F^{II} attains its maximum at the point,

$$s_{max} = \frac{2}{\epsilon||\phi - \psi||_S} \tanh^{-1}(\frac{||\phi - \psi||_S}{C_W - \beta}) + 1,$$

from which we compute

$$F^{II}_{max} \leq 2(C_W + \beta)e^{-\frac{1}{K}\tanh^{-1}(K)} \sinh(\tanh^{-1}(K))$$

where $K = \frac{||\phi-\psi||_S}{C_W-\beta}$. Using the log definition of $\tanh^{-1}$ we arrive at

$$F^{II}_{max} \leq (C_W + \beta)\frac{1+K}{1-K}^{-\frac{1}{2K}} \left(\sqrt{\frac{1+K}{1-K}} - \sqrt{\frac{1-K}{1+K}} \right).$$

For small values of K, say $K < 1/2$ this function is less than $(C_W+\beta)K$. Thus, $||\phi-\psi||_S \leq 2\beta < 1/2(C_W-\beta)$ which implies that $\beta < C_W/5$. Then, we note that

$$||T\phi - T\psi||_S \leq \epsilon C_M(1+\frac{C_W+\beta}{C_W-\beta})e_1(\frac{\epsilon(C_W-\beta)}{2})||\phi-\psi||_S.$$

Again, since C_M and C_W are chosen to be $\mathcal{O}(1)$, then we may choose ϵ sufficiently small to make T a contraction on S, with contraction constant $\mathcal{O}(\epsilon\log(\epsilon))$. Thus, (A.1) has a solution in S and

$$\sup_{t\in[1,\infty)}|W^0(t)-W(t)| = \frac{\mathcal{O}(\epsilon\log(\epsilon))}{1-\mathcal{O}(\epsilon\log(\epsilon))}||W^0-C_W|| = \mathcal{O}(\epsilon^2\log^2(\epsilon)).$$

To justify the use of M^0, we compute

$$|M(t)-M^0(t)| \leq 2C_M e^{-\frac{\epsilon}{2}\int_1^t(W+W^0)ds}\sinh(\frac{C_E\epsilon^2\log(\epsilon)^2}{2})$$

where C_E is an $\mathcal{O}(1)$ constant. Proceeding exactly as above we conclude

$$||M-M^0||_S = \mathcal{O}(\epsilon^2\log(\epsilon)^2).$$

A.3 Proof of Theorem 1

We would like to use the architecture developed in section A.2 to prove this result. An immediate obstacle is that a rescaling of time in the normal form equations (3.16) does not lead to an explicitly solvable equation. Therefore, we must use a modified approach. Namely we consider functions $\phi_1 \in S_1$ with S_1 from (A.2) together with functions $\phi_2 \in S_2$ with

$$S_2 := \left(\phi \in C^0([1,\infty),[(C_M-\alpha)e^{-(C_W+\beta)},(C_M+\alpha)e^{-(C_W-\beta)}]) \quad | \quad \phi_2(1) = C_M\right).$$

We proceed as we did in the justification of the Picard iteration. Namely, we set up an iteration scheme for the full system (3.16) and show that this scheme induces a contraction on the spaces the spaces S_1 and S_2 with contraction constant $\mathcal{O}(\epsilon\log(\epsilon))$. Since the contraction constant goes to zero with ϵ, we may simply use the iteration scheme once with any element $(\phi_1,\phi_2) \in S_1\times S_2$ to derive an asymptotic approximation to the solution. The exact proof will be omitted, except to note that the decay of the $\mathcal{O}(\epsilon^2)$ terms in (3.16) will be exploited along with a number of estimates of terms of the form (A.3).

The proximity of the solution $\Phi(W^0,M^0,t)$ to the true solution follows directly from (3.14).

B Detailed derivation of the Naive Expansion for (4.1).

B.1 The leading order expansion

To obtain $y_1(t)$ we integrate $x_1(t)$ in (4.2) again with zero initial condition:

$$\begin{aligned} y_1(t) &= \int_{t_0}^t\left(A_0 - A_0e^{t_0^2/2}e^{-s^2/2} + B_0e^{t_0^2/2}\int_{t_0}^s e^{-\sigma^2/2}d\sigma - B_0e^{t_0^2/2}e^{-s^2/2}(s-t_0)\right)ds \\ &= A_0(t-t_0) - A_0e^{t_0^2/2}\int_{t_0}^t e^{-s^2/2}ds + B_0e^{t_0^2/2}\int_{t_0}^t\int_{t_0}^s e^{-\sigma^2/2}d\sigma ds \\ &- B_0e^{t_0^2/2}\int_{t_0}^t e^{-s^2/2}(s-t_0)ds. \end{aligned}$$

We now consider the right two terms in the above line and integrate each by parts,

$$\begin{aligned}
I &= B_0 e^{t_0^2/2} \int_{t_0}^{t} \int_{t_0}^{s} e^{-\sigma^2/2} d\sigma ds \\
&= B_0 e^{t_0^2/2} \left[s \int_{t_0}^{s} e^{-\sigma^2/2} d\sigma \right]_{s=t_0}^{t} - B_0 e^{t_0^2/2} \int_{t_0}^{t} s e^{-s^2/2} ds \\
&= B_0 e^{t_0^2/2} t \int_{t_0}^{t} e^{-s^2/2} ds + B_0 e^{t_0^2/2} e^{-t^2/2} - B_0.
\end{aligned}$$

Also,

$$\begin{aligned}
II &= -B_0 e^{t_0^2/2} \int_{t_0}^{t} e^{-s^2/2}(s - t_0) ds \\
&= -B_0 e^{t_0^2/2} \int_{t_0}^{t} s e^{-s^2/2} ds + B_0 e^{t_0^2/2} t_0 \int_{t_0}^{t} e^{-s^2/2} ds \\
&= B_0 e^{t_0^2/2} e^{-t^2/2} - B_0 + B_0 e^{t_0^2/2} t_0 \int_{t_0}^{t} e^{-s^2/2} ds.
\end{aligned}$$

B.2 The second order expansion

The expression for $x_2(t)$ involves an integral with $y_1(t)$ in the integrand. Since y_1 consists of five terms, we split the integral in $x_2(t)$ in five integrals:

$$\begin{aligned}
I &= \int_{t_0}^{t} s e^{s^2/2} A_0 (s - t_0) ds \\
&= A_0 \int_{t_0}^{t} s^2 e^{s^2/2} ds - A_0 t_0 \int_{t_0}^{t} s e^{s^2/2} ds \\
&= A_0 \left[s e^{s^2/2} \right]_{s=t_0}^{t} - A_0 \int_{t_0}^{t} e^{s^2/2} ds - A_0 t_0 \left[e^{s^2/2} \right]_{s=t_0}^{t} \\
&= A_0 t e^{t^2/2} - A_0 t_0 e^{t_0^2/2} - A_0 \int_{t_0}^{t} e^{s^2/2} ds - A_0 t_0 e^{t^2/2} + A_0 t_0 e^{t_0^2/2},
\end{aligned}$$

$$\begin{aligned}
II &= -\int_{t_0}^{t} s e^{s^2/2} A_0 e^{t_0^2/2} \int_{t_0}^{s} e^{-\sigma^2/2} d\sigma ds \\
&= -A_0 e^{t_0^2/2} \int_{t_0}^{t} s e^{s^2/2} \int_{t_0}^{s} e^{-\sigma^2/2} d\sigma ds \\
&= -A_0 e^{t_0^2/2} \left[e^{s^2/2} \int_{t_0}^{s} e^{-\sigma^2/2} d\sigma \right]_{s=t_0}^{t} + A_0 e^{t_0^2/2} \int_{t_0}^{t} e^{s^2/2} e^{-s^2/2} ds \\
&= -A_0 e^{t_0^2/2} e^{t^2/2} \int_{t_0}^{t} e^{-s^2/2} ds + A_0 e^{t_0^2/2} (t - t_0),
\end{aligned}$$

$$
\begin{aligned}
III &= \int_{t_0}^{t} se^{s^2/2}B_0e^{t_0^2/2}(s+t_0)\int_{t_0}^{s} e^{-\sigma^2/2}d\sigma ds \\
&= B_0e^{t_0^2/2}\int_{t_0}^{t} s^2e^{s^2/2}\int_{t_0}^{s} e^{-\sigma^2/2}d\sigma ds + B_0t_0e^{t_0^2/2}\int_{t_0}^{t} se^{s^2/2}\int_{t_0}^{s} e^{-\sigma^2/2}d\sigma ds \\
&= B_0e^{t_0^2/2}\left[se^{s^2/2}\int_{t_0}^{s} e^{-\sigma^2/2}d\sigma\right]_{s=t_0}^{t} - B_0e^{t_0^2/2}\int_{t_0}^{t} e^{s^2/2}\int_{t_0}^{s} e^{-\sigma^2/2}d\sigma ds \\
&- B_0e^{t_0^2/2}\int_{t_0}^{t} se^{s^2/2}e^{-s^2/2}ds \\
&+ B_0t_0e^{t_0^2/2}\left[e^{s^2/2}\int_{t_0}^{s} e^{-\sigma^2/2}d\sigma\right]_{s=t_0}^{t} - B_0t_0e^{t_0^2/2}\int_{t_0}^{t} e^{s^2/2}e^{-s^2/2}ds \\
&= B_0e^{t_0^2/2}te^{t^2/2}\int_{t_0}^{t} e^{-s^2/2}ds - B_0e^{t_0^2/2}\int_{t_0}^{t} e^{s^2/2}\int_{t_0}^{s} e^{-\sigma^2/2}d\sigma ds - B_0e^{t_0^2/2}\left(\frac{t^2}{2}-\frac{t_0^2}{2}\right) \\
&+ B_0t_0e^{t_0^2/2}e^{t^2/2}\int_{t_0}^{t} e^{-s^2/2}ds - B_0t_0e^{t_0^2/2}(t-t_0),
\end{aligned}
$$

$$
\begin{aligned}
IV &= \int_{t_0}^{t} se^{s^2/2}2B_0e^{t_0^2/2}e^{-s^2/2}ds \\
&= 2B_0e^{t_0^2/2}\int_{t_0}^{t} sds \\
&= B_0e^{t_0^2/2}(t^2-t_0^2),
\end{aligned}
$$

$$
\begin{aligned}
V &= -\int_{t_0}^{t} se^{s^2/2}2B_0ds \\
&= -2B_0\int_{t_0}^{t} se^{s^2/2}ds \\
&= -2B_0(e^{t^2/2}-e^{t_0^2/2}).
\end{aligned}
$$

Multiplying expressions I through V by $e^{-t^2/2}$ we arrive at the solution for x_2, (4.5).

Now, to find $y_2(t)$ we integrate $x_2(t)$ with zero initial conditions,

$$
y_2(t) = \int_{t_0}^{t} x_2(s)ds.
$$

Thus,

$$
\begin{aligned}
y_2(t) &= A_0\left(\frac{t^2}{2}-\frac{t_0^2}{2}\right) - A_0 t_0 e^{t_0^2/2}\int_{t_0}^{t} e^{-s^2/2}ds - A_0\int_{t_0}^{t} e^{-s^2/2}\int_{t_0}^{s} e^{\sigma^2/2}d\sigma ds \\
&- A_0 t_0(t-t_0) + A_0 t_0 e^{t_0^2/2}\int_{t_0}^{t} e^{-s^2/2}ds - A_0 e^{t_0^2/2}\int_{t_0}^{t}\int_{t_0}^{s} e^{-\sigma^2/2}d\sigma ds \\
&+ A_0 e^{t_0^2/2}\int_{t_0}^{t} s e^{-s^2/2}ds - A_0 t_0 e^{t_0^2/2}\int_{t_0}^{t} e^{-s^2/2}ds + B_0 e^{t_0^2/2}\int_{t_0}^{t} s\int_{t_0}^{s} e^{-\sigma^2/2}d\sigma ds \\
&- B_0 e^{t_0^2/2}\int_{t_0}^{t} e^{-s^2/2}\int_{t_0}^{s} e^{\sigma^2/2}\int_{t_0}^{\sigma} e^{-\tau^2/2}d\tau d\sigma ds \\
&- B_0 e^{t_0^2/2}\int_{t_0}^{t}\frac{s^2}{2}e^{-s^2/2}ds + B_0 e^{t_0^2/2}\frac{t_0^2}{2}\int_{t_0}^{t} e^{-s^2/2}ds + B_0 t_0 e^{t_0^2/2}\int_{t_0}^{t}\int_{t_0}^{s} e^{-\sigma^2/2}d\sigma ds \\
&- B_0 t_0 e^{t_0^2/2}\int_{t_0}^{t} s e^{-s^2/2}ds + B_0 t_0^2 e^{t_0^2/2}\int_{t_0}^{t} e^{-s^2/2}ds + B_0 e^{t_0^2/2}\int_{t_0}^{t} s^2 e^{-s^2/2}ds \\
&- B_0 t_0^2 e^{t_0^2/2}\int_{t_0}^{t} e^{-s^2/2}ds - 2B_0(t-t_0) + 2B_0 e^{t_0^2/2}\int_{t_0}^{t} e^{-s^2/2}ds.
\end{aligned}
$$

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