Math 214
Lecture 9

Classification of equilibria of $y' = f(y)$:

1. $y = a$ is stable equilibrium
   - Trajectories are attracted from both sides

2. $y = a$ is unstable equilibrium
   - Trajectories are repelled

3. Both of these give $y = a$ as a semistable equilibrium

Example: $y' = 2y(y-2)$

- Unstable $y = 2$
- Stable $y = 0$. 

Example 2. \( y' = 2y^2(y-2) \)
Classify equilibria.

1) \( y=0 \) equilibria
\( y=2 \) equilibria

In general, \( f(y) = A(y-a)^d(y-b)^e \) etc.

if \( d \)-odd, then at \( y=a \) the graph (changing sign) crosses the \( y \)-axis
if \( b \)-even, then at \( y=b \) the graph (does not change sign) touches the \( y \)-axis.

if e.g. \( d = 3 \), \( b = 10 \) \( \Rightarrow \ f(y) = -3(y-a)^3(y-b)^10 \)
\( A = -3 \)
\( a = 1 \) \( b = 2 \)

\( f(y) = -3(y-1)^3(y-2)^10 \)

If \( f(y) = y(y-1)^5(2-y)^2 \)

2) \( y \)
y = 2 unstable
\( y = 0 \) semistable

If \( y(0) = 1 \) \( \lim_{t \to \infty} y(t) = 0 \)
If \( y(0) = 2 \) \( \lim_{t \to \infty} y(t) = 2 \)
Ex. \[ \frac{dy}{dt} = y^2 (3-2y)(5+y)^3 \]

Find \( \lim_{t \to 0} y(t) \) if \( y(0) = \frac{1}{2} \).

Rewrite \[ \frac{dy}{dt} = -2y^2(y - \frac{3}{2})(5+y)^3 \]

\[ y = -5 \quad 0 \quad \frac{3}{2} \]

Test \( y = 2 \) to see the sign.

\( (5+y)^3 \)
\( y^2 \)
\( y - \frac{3}{2} \)

\( \lim_{t \to \infty} y(t) = \frac{3}{2} \) for \( y(0) = \frac{1}{2} \)

For which \( y(0) \) will the trajectory become unbounded?

Only for \( y(0) < -5 \)

Inflection points analysis:

Inflection pt corresponds to \( y'' = 0 \)

\[ y' = \frac{dy}{dt} = f(y) \Rightarrow y'' = f'(y) \cdot y' = f' \cdot f \]

\[ \frac{d^2y}{dt^2} = \frac{af}{dy} \cdot \frac{dy}{dt} \text{ chain rule} \]

\( y'' = 0 \) when \( f = 0 \), or \( f' = 0 \)

Equilibria condition to not inflection pts have an inflection pt.

If \( f' \cdot f > 0 \) we have concave up trajectory.

If \( f' \cdot f < 0 \) we have concave down.

In Ex. 1: \[ \frac{dy}{dt} = 2y(y-2) \]

\( f(y) = 2y(y-2) = 2y^2 - 4y \)

\( f'(y) = 4y - 4 = 0 \) \( \boxed{y = 1} \) is the inflection point
<table>
<thead>
<tr>
<th>Signs</th>
<th>$f$</th>
<th>$f'$</th>
<th>$f \cdot f'$</th>
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<tbody>
<tr>
<td>$y &lt; 0$</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$0 &lt; y &lt; 1$</td>
<td>-</td>
<td>-</td>
<td>+</td>
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<tr>
<td>$1 &lt; y &lt; 2$</td>
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<td>-</td>
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<tr>
<td>$y &gt; 2$</td>
<td>+</td>
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$f = 2y(y-2) \quad f' = 4(y-1)$

Ex. 3. $y' = (y+1)(y-4)$

Classify equilibria and find inflection pts. if any.

Ex. 4. $y' = -30(y-1)^{100}(2-y)^{35}(y-25)^{-3}$

? = $\lim_{t \to 0} y(t)$ when $y(0) = 10$