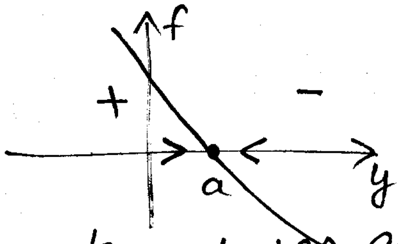
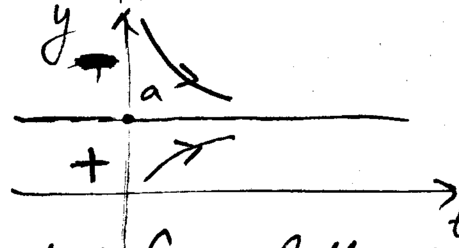
Math 214.

Lecture 9

Classification of equilibria of $y' = f(y)$:

(1)

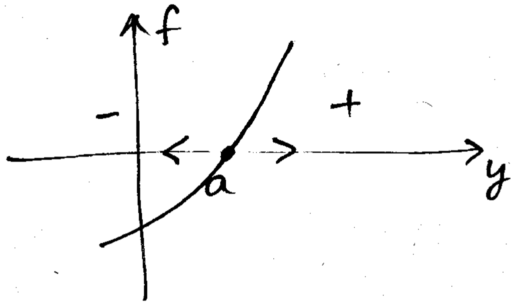


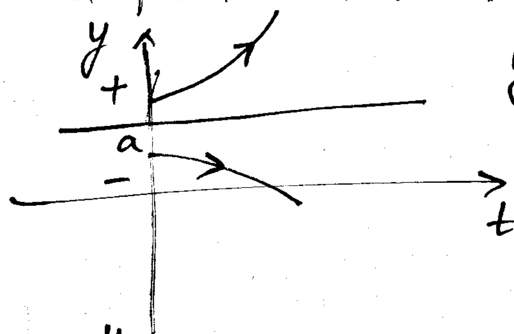


$y=a$ is asymp. stable equilibrium

trajectories are attracted from both sides

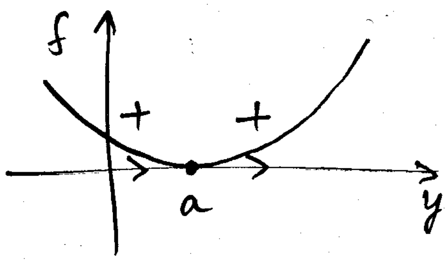
(2)

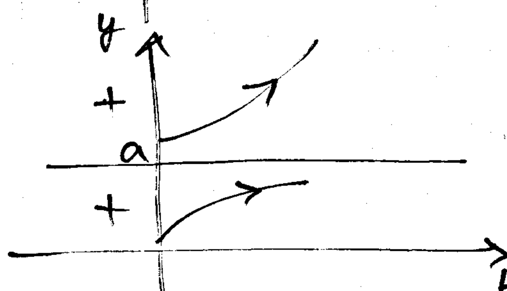


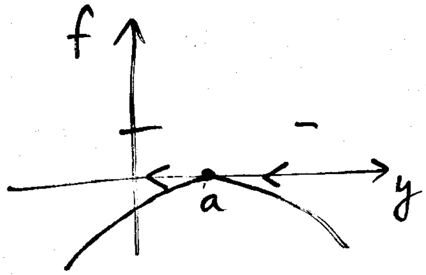


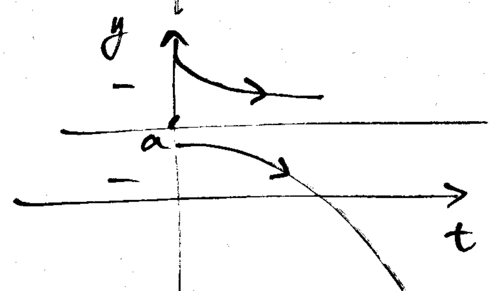
$y=a$ is unstable equilibrium

(3)



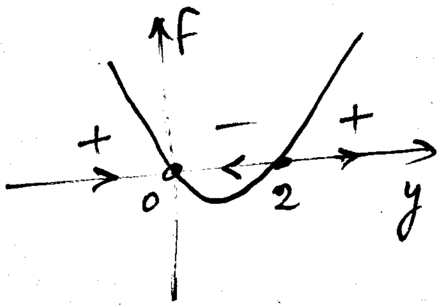


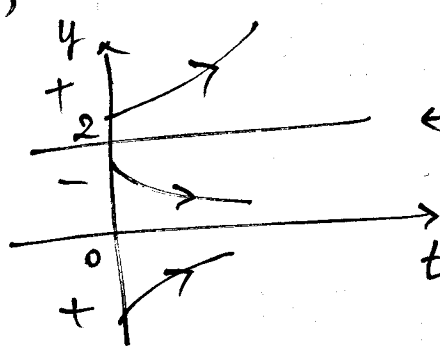




both of these give $y=a$ as a semistable equilibrium

Example $y' = 2y(y-2)$





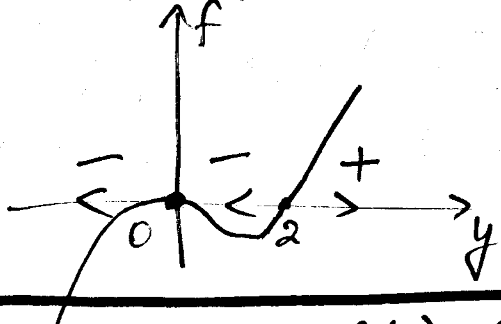
← unstable $y=2$

← stable $y=0$.

Example 2. $y' = 2y^2(y-2)$

Classify equilibria.

1)



$y=0$, $y=2$ equilibria

Side note.

In general, $f(y) = A(y-a)^{\alpha}(y-b)^{\beta}$ etc.

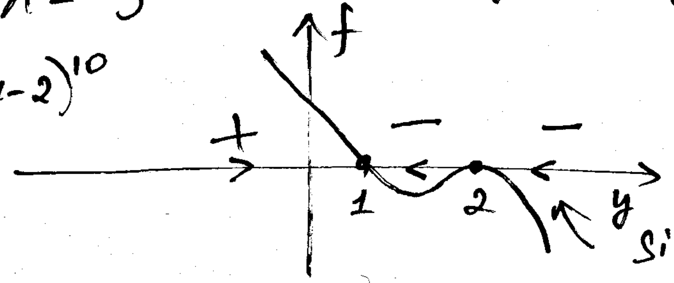
if α-odd, then at $y=a$ the graph (changing sign) crosses the y-axis

if β-even, then at $y=a$ the graph (does not change sign) touches the y-axis.

if e.g. $\alpha = 3$, $\beta = 10$, $A=-3$ $\Rightarrow$ $f(y) = -3(y-a)^3(y-b)^{10}$

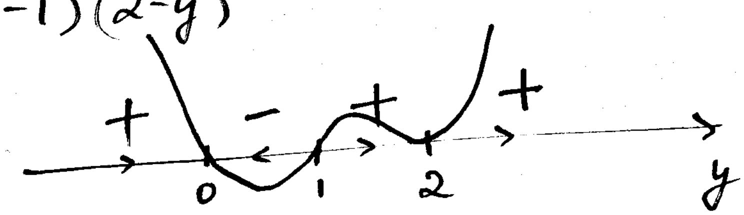
$a=1$ $b=2$

$f(y) = -3(y-1)^3(y-2)^{10}$

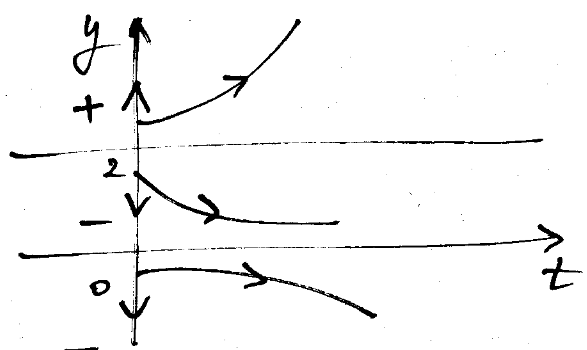


since $f(y)<0$ when $y>2$

If $f(y) = y(y-1)^5(2-y)^2$



2)



$y=2$ unstable

$y=0$ semistable

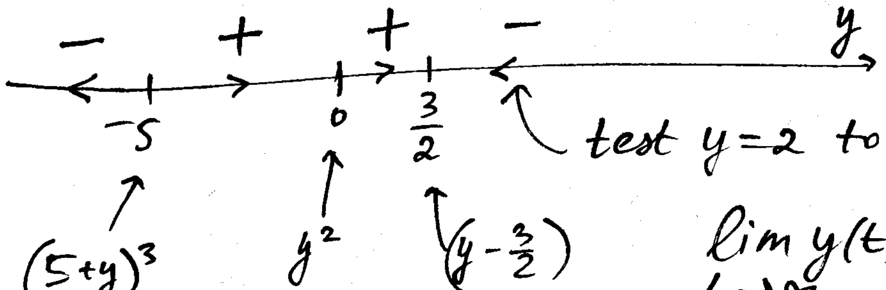
If $y(0)=1$ $\lim_{t\to\infty} y(t) = 0$

If $y(0)=2$ $\lim_{t\to\infty} y(t) = 2$

Ex. $\frac{dy}{dt} = y^2(3-2y)(5+y)^3$

Find $\lim_{t\to\infty} y(t)$ if $y(0) = \frac{1}{2}$.

Rewrite $\frac{dy}{dt} = -2y^2(y-\frac{3}{2})(5+y)^3$



test $y=2$ to see the sign

$\lim_{t\to\infty} y(t) = \frac{3}{2}$ for $y(0) = \frac{1}{2}$

For which $y(0)$ will the trajectory become unbounded?

Only for $y(0) < -5$

Inflection points analysis:

Inflection pt corresponds to $y'' = 0$

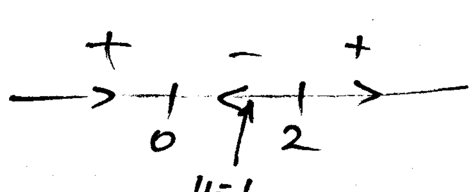
$y' = \frac{dy}{dt} = f(y) \Rightarrow y'' = f'(y)\cdot y' = f' \cdot f$

$\frac{d^2y}{dt^2} = \frac{df}{dy} \cdot \frac{dy}{dt}$ chain rule

$y'' = 0$ when $\underbrace{f = 0}_{\text{equilibria not inflection pts}}$, or $\boxed{f' = 0}$ Condition to have an inflection pt.

If $f' \cdot f > 0$	we have concave up trajectory
If $f' \cdot f < 0$	we have concave down.

In Ex. 1: $\frac{dy}{dt} = 2y(y-2)$

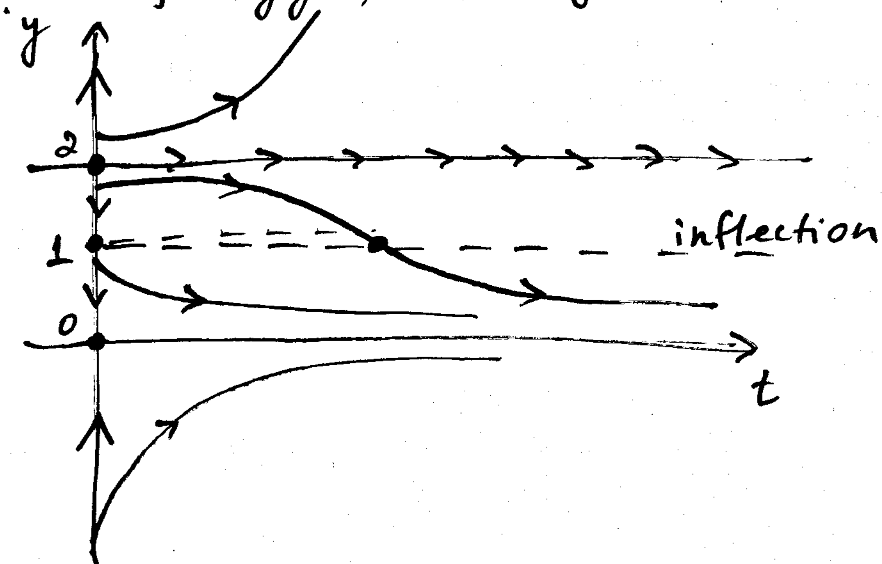


$f(y) = 2y(y-2) = 2y^2 - 4y$

$f'(y) = 4y - 4 = 0$ $\boxed{y=1}$ is the inflection point

Signs	f	f'	$f \cdot f'$	
$y<0$	$+$	$-$	$-$	concave down
$0<y<1$	$-$	$-$	$+$	up
$1<y<2$	$-$	$+$	$-$	down
$y>2$	$+$	$+$	$+$	up

$f = 2y(y-2) \quad f' = 4(y-1)$



Ex.3. $y' = (y+1)(y-4)$

Classify equilibria and find inflection pts (if any).

Ex.4. $y' = -30(y-1)^{100}(2-y)^{35}(y-25)^{-3}$

$? = \lim_{t\to\infty} y(t)$ when $y(0)=10$