Math 214.
Lecture 8

Last time:

Thm 1: for linear 1st order eqn $\begin{cases} y' + p(t)y = g(t) \\ y(t_0) = y_0 \end{cases}$

IVP has a unique solution in any interval containing y_0 provided $p(t), g(t)$ are continuous.

Thm 2. for nonlinear 1st order eqn $\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$

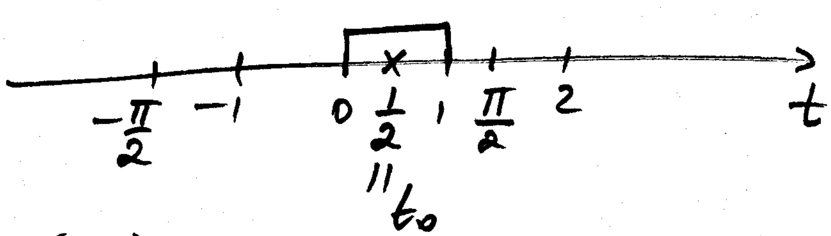
IVP has a unique sol. in any region in (t,y) plane containing (t_0, y_0) provided $f(t,y)$ and $\frac{\partial f}{\partial y}(t,y)$ are continuous.

Ex. $\begin{cases} t^2(\cos t)\, y' + (\sqrt{t})\cdot y - \frac{1}{t^2-1} = 0 \\ y\left(\frac{1}{2}\right) = 0 \end{cases}$

Find maximal interval where solution exists.

Linear eqn $\Rightarrow$ Thm 1 $\Rightarrow$ standard form

$$y' + \underbrace{\frac{\sqrt{t}}{t^2(\cos t)}}_{p(t)} \cdot y = \underbrace{\frac{1}{t^2(t^2-1)\cos t}}_{g(t)}$$



$p(t): \begin{cases} t \neq 0 \\ t \neq \frac{\pi}{2} + \pi k \end{cases}$

$g(t): \begin{cases} t \neq 0 \\ t \neq 1,\ t \neq -1 \\ t \neq \frac{\pi}{2} + \pi k \end{cases}$

$I = (0, 1)$ — max interval where solution is guaranteed to exist, and be unique.

Ex. $\begin{cases} y' + \sqrt{ty} = \dfrac{\cos t}{t^2-1} \\ y(t_0) = y_0 \end{cases}$ Find all intervals where solution is unique

$\sqrt{ty}$: nonlinear term $\Rightarrow$ Thm 2 $\Rightarrow y' = f(t,y)$

$$y' = \underbrace{\frac{\cos t}{t^2-1} - \sqrt{ty}}$$

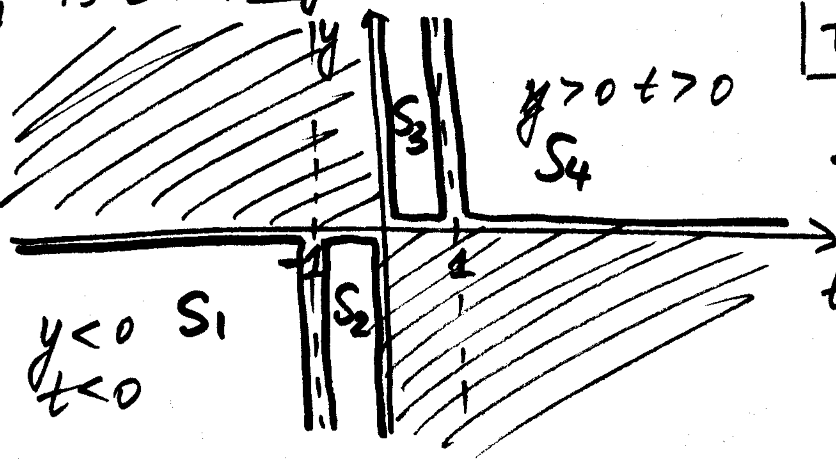
$$f(t,y) = \frac{\cos t}{t^2-1} - (ty)^{\frac{1}{2}} \Rightarrow \boxed{t \neq 1,\ t \neq -1 \\ t \cdot y \geq 0}$$

$\uparrow$ f is cts

$$\frac{\partial f}{\partial y} = \underbrace{-\frac{1}{2}(ty)^{-\frac{1}{2}} \cdot t}$$ by chain rule

$\frac{\partial f}{\partial y}$ is cts if $\boxed{ty > 0}$

$\boxed{t \neq 1, -1,\ ty > 0}$



S_1, S_2, S_3, S_4 are all regions where Sol. of IVP can be unique

If we add information of the type $y(-\frac{1}{2}) = -2$

$S_2 = \{(t,y) \mid -1 < t < 0,\ y < 0\}$ is the maximal regions where solution is unique,

though it will exist in $\{(t,y) \mid -1 < \underline{t \leq 0}, \underline{y \leq 0}\}$

where f is cts but $\frac{\partial f}{\partial y}$ is not.

§2.5 Autonomous equations.

$\frac{dy}{dt} = f(y) \leftarrow$ no time dependence in the r.h.s.

These equations are always separable.

Ex. $\frac{dy}{dt} = y^2(3-2y)(5+y)^3$ $\quad \lim_{t \to \infty} y(t) = ?$

Can be answered without solving the eqn.

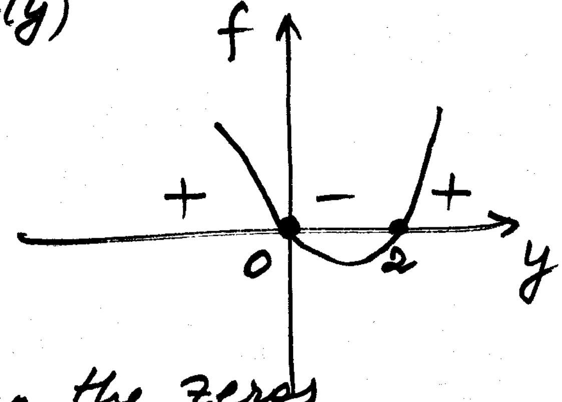
Qualitative analysis of autonomous eqns

Simple example. $\frac{dy}{dt} = \underbrace{2y(y-2)}_{f(y)}$

Step 1. Graph $f(y)$:

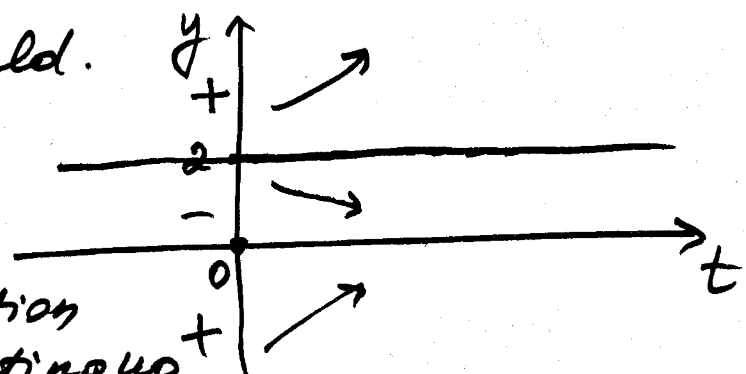
1) locate zeros of $f(y)$

$$2y(y-2)=0$$
$$\downarrow \quad \downarrow$$
$$0 \quad 2$$



2) find signs of $f(y)$ between the zeros

Step 2. Transfer this information ~~to~~ to the direction field.



$\left.\begin{matrix} y=0 \\ y=2 \end{matrix}\right\}$ equilibrium solutions

$f(y)>0$ means direction field in pointing up

$f(y)<0$ - - - - down

$$\lim_{t\to\infty} y(t) = \begin{cases} +\infty, & \text{if } y_0 > 2 \\ 0, & \text{if } 0 < y_0 < 2 \\ & \text{or } y_0 \le 0 \\ 2, & \text{if } y_0 = 2 \end{cases}$$