Math 214.
Lecture 7

Nonlinear vs. linear equation

Thm 1. (Linear problem).

$$\begin{cases} y' + p(t)y = g(t) \quad (*) \\ y(t_0) = y_0 \end{cases}$$

If $p(t), g(t)$ are continuous in some interval $I: \alpha < t < \beta$ containing $t = t_0$ there there is a unique solution to $(*)$ for any t in I and any y_0.

Pf: $$y(t) = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)\, g(s)\, ds + y_0\right] \leftarrow \text{solution to IVP } (*)$$

intergrable, continuous fct

$$\mu(t) = e^{\int p(t)dt}$$

If the integral formula exists, then it provides the solution of $(*)$ which is unique.

Thm 2 (Nonlinear problem).

$$\begin{cases} y' = f(t,y) \quad (**) \\ y(t_0) = y_0 \end{cases}$$

If $f(t,y)$ and $\frac{\partial f}{\partial y}(t,y)$ are continuous in region $R: \alpha < t < \beta, \gamma < y < \delta$ containing (t_0, y_0) then there exist a unique solution to $(**)$ in some interval $t_0 - h < t < t_0 + h$ contained inside $\alpha < t < \beta$.

Remarks:

1) If in Thm 2, $f(t,y) = -p(t)y + g(t)$
(this gives $y' = -p(t)y + g(t)$)
$y' + p(t)y = g(t)$

$$\left.\begin{array}{l} f - (\text{cts}) \leftarrow \text{continuous} \\ \frac{\partial f}{\partial y} = -p(t) \end{array}\right\} \Rightarrow$$ both $p(t), g(t)$ have to be continuous for existence and uniqueness

Thm 2 reduces to Thm 1 in this case.

2) f is continuous in Thm 2, it is enough to guarantee existence (but not uniqueness).

3) If Thm 1 or Thm 2 holds, solutions of ⊛ or ⊛⊛ cannot intersect each other.

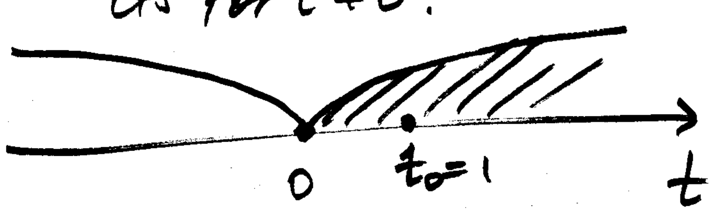
Ex. 1 $\begin{cases} tx' = x + 3t^2 \\ x(1) = 2 \end{cases}$

Is there a solution to this IVP and is it unique? Specify where solution is valid.
(OR: find the maximal interval where solution exists and is unique).

$$x' = \frac{x + 3t^2}{t} = \left(\frac{1}{t}\right)x + 3t \quad \leftarrow \text{linear}$$

$$p(t) = -\frac{1}{t}, \quad g(t) = 3t$$

cts for $t \neq 0$. cts everywhere



Solution is unique in $\boxed{(0, \infty)}$.
by Thm 1

Ex. 2 $\begin{cases} x' = (x-1)\cos(xt) \\ x(0) = 1 \end{cases}$ — nonlinear

Question: can be find solution to this IVP without integrating.

$(**)\begin{cases} x' = f(t,x) \\ x(t_0) = x_0 \end{cases}$ $\begin{cases} f(t,x) = (x-1)\cos(xt) \\ \dfrac{\partial f}{\partial x} = \cos(xt) - t(x-1)\sin(xt) \end{cases}$

are continuous for all (t,x).

By Thm 2, solution exists and is unique everywhere.

Pick $x \equiv 1$. It satisfies $x(0)=1$

and $x' = 0 = (x-1)\cdot\cos(xt)$ so $x \equiv 1$ solves IVP

Since Solution is unique, this is the answer.

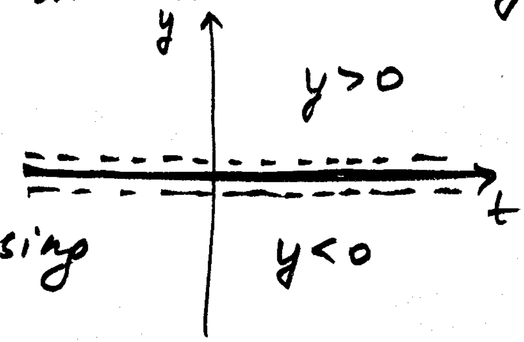
Ex. 3. $\begin{cases} y' = y^{1/3} \\ y(0) = 0 \end{cases}$

Nonlinear problem $\Rightarrow$ apply Thm 2.

$f(t,y) = y^{1/3}$, $\dfrac{\partial f}{\partial y} = \frac{1}{3} y^{-2/3}$

cts everywhere — discontinuous at $y=0$.

Since $y(0) = \underline{0}$ Thm 2 cannot guarantee uniqueness, only existence of the solution passing through $(0,0)$.



Let's solve this IVP:

$$\frac{dy}{dt} = y^{1/3} \Rightarrow \int \frac{dy}{y^{1/3}} = \int dt$$

$\frac{3}{2} y^{2/3} = t + C \leftarrow$ gen. solution

Since $y(0)=0 \Rightarrow C=0$.

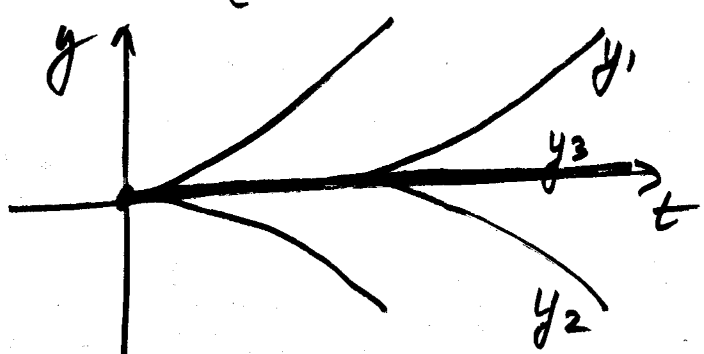
$\Rightarrow y^{2/3} = \frac{2}{3} \cdot t \Rightarrow y = \left(\frac{2}{3}t\right)^{3/2}$ – solution to IVP.

$y = -\left(\frac{2}{3}t\right)^{3/2}$ – another solution to IVP.

In fact, $y \equiv 0$ is missed by this solution family since it was divided by in the beginning.

Solution to IVP looks like this:

$$y = \begin{cases} 0, & 0 \le t < t_0 \\ \pm\left(\frac{2}{3}t\right)^{3/2}, & t \ge t_0. \end{cases}$$



non-unique solution
no contradiction with Thm 2.

Linear	Nonlinear
① discontinuities of solution happen only at discontinuities of $p(t)$ or $g(t)$.	① Solution can be discontinuous where $f(t,y)$ is continuous.
② Always have general solution with an arbitrary constant that gives all solutions.	② Usually "general solution" does not exist.