Math 214.
Lecture 6

Mixing problem.

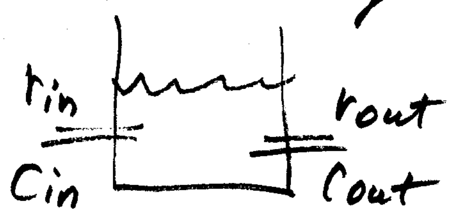
brine solution in a tank

rate in = 3 gal/min

rate out = 2 gal/min, initially 300 gal of solution $\quad V_0 = 300$

50 lbs of salt dissolved at $t=0$

inflowing concentration = 2 lb/gal



$Q(t)$ = amount of salt at time t

$V(t)$ = amount of solution at time t

$$\frac{dQ}{dt} = \underline{r_{in}}\,\underline{C_{in}} - \underline{r_{out}}\,C_{out}, \quad C_{out} = \frac{Q(t)}{V(t)}$$

$$\frac{dQ}{dt} = r_{in}C_{in} - r_{out}\cdot\frac{Q(t)}{V(t)}, \quad V(t) = V_0 + (r_{in} - r_{out})t$$

$$V(t) = 300 + t$$

$$\Rightarrow \boxed{\frac{dQ}{dt} = 3\cdot 2 - 2\cdot\frac{Q}{300+t}, \quad Q(0) = 50} \text{ IVP}$$

$$\frac{dQ}{dt} = 6 - \frac{2Q}{300+t}$$

$$\frac{dQ}{dt} + \underbrace{\frac{2}{300+t}}_{p(t)}\cdot Q = 6, \quad \mu(t) = e^{\int p(t)dt} =$$

$$= e^{\int \frac{2\,dt}{t+300}} = e^{2\ln|t+300|}$$

$$\boxed{\mu(t) = (t+300)^2}$$

$$\Rightarrow [\mu(t)\cdot Q(t)]' = 6\mu(t)$$

$$\left[(t+300)^2 Q(t)\right]' = 6(t+300)^2$$

$$(t+300)^2 Q(t) = 6\int (t+300)^2 dt + C$$

$$(t+300)^2 Q(t) = 6 \cdot \frac{(t+300)^3}{3} + C \qquad \begin{array}{l} u = t+300 \\ du = dt \end{array}$$

$$\boxed{Q(t) = 2(t+300) + C(t+300)^{-2}} \text{ gen. sol.}$$

What is $Q(t)$ after a $\underset{t\to\infty}{\underline{\text{long time}}}$?

$$\lim_{t\to\infty} Q(t) = \lim_{t\to\infty} 2(t+300) = \infty$$

if $V_{max} = 500 \text{ (gal)}$,

$$V(t) = 300 + t \leq 500$$

$$V(t_{overflow}) = 300 + t_{overflow} = 500$$

$$\Rightarrow t_{overflow} = 200 \text{(mins)}.$$

Amount of salt at time of overflow is $\underline{Q(200)}$.

Solve for C: $Q(0) = 50 \Rightarrow$

$$50 = 2\cdot 300 + C \cdot 300^{-2}$$

$$50 \cdot 300^2 = 2 \cdot 300^3 + C$$

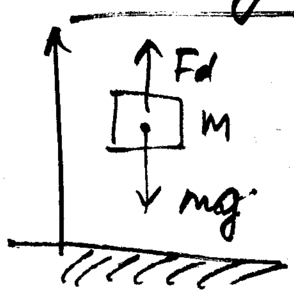
$$C = 300^2(50 - 2\cdot 300) =$$

$$= -300^2 \cdot 550$$

$$Q(200) = 2\cdot 500 - 550 \cdot 300^2 \cdot 500^{-2}$$

$$= 1000 - 550\left(\frac{3}{5}\right)^2 = \frac{25000 - 9\cdot 550}{25}$$

Falling body problem.



Body falling in atmosphere (offering resistance) with velocity $v(t)$.

$F_d = \gamma |v|$ – usual form of drag force

Newton's law: $F_{net} = m\vec{a}$

a = acceleration

$\boxed{v'(t) = a(t)}$

$x(t)$ - position $\Rightarrow$ $\boxed{x'(t) = v(t)}$

$\vec{F}_{net} = m\vec{g} + \vec{F}_d$

In coordinates: $F_{net} = mg - \gamma|v|$

$$\Rightarrow \begin{cases} mg - \gamma v = mv' \\ v(0) = v_0 \end{cases} \Leftarrow \text{DE for body falling down}$$

Ex. Ball of mass 0.15 kg, thrown upward with initial velocity 20 m/sec from roof of a building 30m high. Neglect air resistance. What is the max height it will reach?

$$mv' = mg \Rightarrow \begin{cases} v' = g \\ v(0) = v_0 \end{cases} \qquad \begin{aligned} v(t) &= -gt + C \\ v(0) &= C = v_0 \end{aligned}$$

$$\Rightarrow v(t) = v_0 - gt$$

It reaches max height when $v(T) = 0$

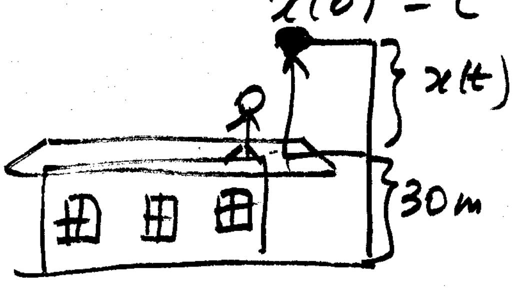
i.e. $v(T) = v_0 - gT = 0 \Rightarrow \boxed{T = \frac{v_0}{g}}$ (sec)

$x'(t) = v(t)$

$$\Rightarrow x(t) = \int v(t)\,dt = \int (v_0 - gt)\,dt = v_0 t - \frac{gt^2}{2} + C$$

$x(t=0) = 30(m) \Leftarrow$ roof of the building

$$x(0) = C = 30 \Rightarrow x(t) = \overset{v_0}{20}t - \frac{gt^2}{2} + 30$$



[position at time t before we stop]

$$\boxed{x(t) = v_0 t - \frac{gt^2}{2} + 30}$$

Max height is $x(T) = \overset{v_0}{\cancel{20}} \cdot \frac{v_0}{g} - \frac{g \cdot v_0^2}{g^2 \cdot 2} + 30$

$$x(T) = \frac{v_0^2}{g} - \frac{v_0^2}{2g} + 30 = \frac{v_0^2}{2g} + 30$$

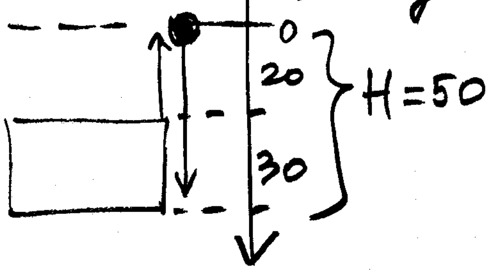
$$= \frac{20^2}{2 \cdot 10} + 30 = 50\,(m)$$

$\underline{\text{max height}}$

Now find the time when the ball hits the ground.

$mv' = mg$ (since starting at H_{max})

$v' = g \Rightarrow v = gt + C$ $\quad v(0) = 0 \Rightarrow$



$v = gt$

$\Rightarrow x = \int gt = \frac{gt^2}{2} + C$

$x(0) = 0$

Find $x(T) = 50 \Rightarrow \frac{gT^2}{2} = 50$

If $g = 10 \frac{m}{sec^2} \Rightarrow T^2 = 10$

$\boxed{T = \sqrt{10}\,(sec)}$