Math 214.

Lecture 5.

Ex. 1 $\begin{cases} y' = \dfrac{x}{1+2y} \\ y(-1) = 0 \end{cases}$

$$(1+2y)\,dy = x\,dx$$

$$y + y^2 = \frac{x^2}{2} + C \Leftarrow y(-1) = 0$$

$$0 + 0 = \frac{1}{2} + C \Rightarrow C = -\frac{1}{2}$$

$y^2 + y - \left(\frac{x^2}{2} - \frac{1}{2}\right) = 0$ implicit form of sol.

Apply quadratic formula: $ay^2 + by + c = 0$

$$y = \frac{-1 \pm \sqrt{1 + 4\left(\frac{x^2}{2} - \frac{1}{2}\right)}}{2} = \frac{-1 \pm \sqrt{2x^2 - 1}}{2}$$

Have to choose a branch satisfying $y(-1) = 0$.

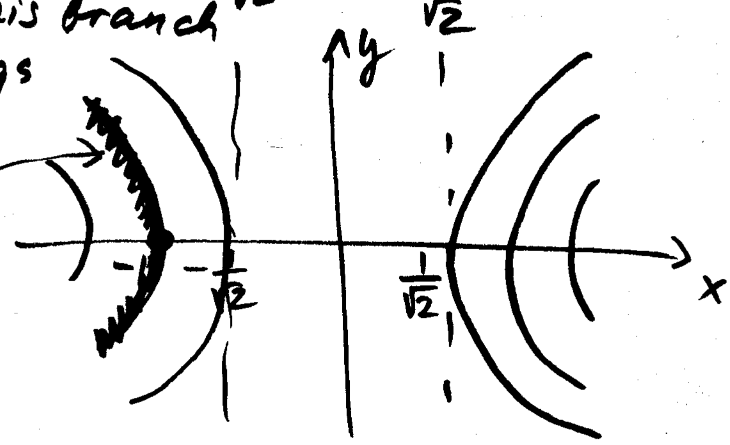
$$0 = y(-1) = \frac{-1 \pm \sqrt{2(-1)^2 - 1}}{2} = \frac{-1 \pm 1}{2} = \begin{cases} 0, \oplus \leftarrow \text{works} \\ -1, \ominus \leftarrow \text{does not work} \end{cases}$$

$\Rightarrow$ Final answer:

$$y(x) = \frac{-1 + \sqrt{2x^2 - 1}}{2}$$

Domain: $2x^2 - 1 \geq 0 \Rightarrow |x| \geq \frac{1}{\sqrt{2}}$

this
where this branch belongs



Ex.2 $\begin{cases} y' = \dfrac{1-2x}{y} \\ y(1) = -2 \end{cases}$

$$\int y\,dy = \int (1-2x)\,dx$$

$\frac{y^2}{2} = x - x^2 + C$ $\boxed{\text{gen. sol.}}$

Plug in $y(1) = -2 \Rightarrow \frac{(-2)^2}{2} = 1 - 1^2 + C$

$\boxed{2 = C}$

$$\Rightarrow \frac{y^2}{2} = x - x^2 + 2$$

$$y^2 = 2x - 2x^2 + 4$$

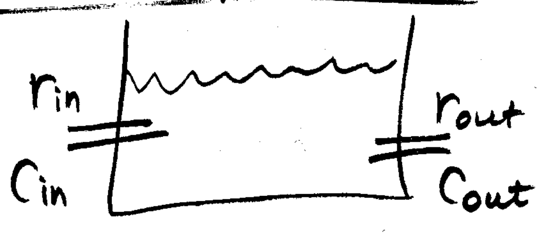
$$y = \pm\sqrt{2x - 2x^2 + 4}$$

Since $y(1) = -2$, we pick $\ominus$ $\Rightarrow y = -\sqrt{2x - 2x^2 + 4}$ explicit solution of IVP

§2.3. Modeling.

Growth models : $\begin{cases} \frac{dx}{dt} = kx \\ x(0) = x_0 \end{cases}$ were already discussed $\rightarrow x(t) = x_0 e^{kt}$

Mixture problems:



C_{in} - concentration of the substance flowing into the tank

C_{out} - concentr. of the Solution ~~at~~ in the outflowing pipe

$V(t)$ = volume of solution at time t

V_{max} = tank capacity

$Q(t)$ = amount of salt in tank at time t

can be any other substance

$$\begin{cases} \dfrac{dQ}{dt} = \underline{r_{in}}\,\underline{C_{in}} - \underline{r_{out}}\,C_{out} \\ Q(0) = Q_0 \end{cases}$$

C_{out} ← ?

Q_0 ← initial amount of salt in the tank

r_{in}, r_{out}, C_{in} – known

C_{out} – you have to find before solving for $Q(t)$.

Ex. Large tank holds 300 gal of brine solution. A brine solution is being pumped in at a rate of 3 gal/min, ← mixes with the brine inside and is pumped out at the same rate. The concentration of salt in the inflowing solution is 2 lb/gal.

Q: If 50 lbs of salt were dissolved initially, how much salt is there after a long time?

$V(0) = 300$ gal (initial volume of solution)

$r_{in} = 3$ gal/min , C_{out} = concentration inside the tank

$r_{out} = r_{in}$

$C_{in} = 2$ (lb/gal)

$Q(0) = 50$ (lbs)

$$\Downarrow$$

$$C_{out} = \frac{Q(t)}{V(t)} = \frac{Q(t)}{300}$$

$$V(t) = V(0) + (r_{in} - r_{out})t$$

$$\Rightarrow \left(\frac{lb}{min}\right) \frac{dQ}{dt} = 3\left(\frac{\cancel{gal}}{min}\right) \cdot 2\left(\frac{lb}{\cancel{gal}}\right) - 3\left(\frac{\cancel{gal}}{min}\right) \cdot \frac{Q}{300}\left(\frac{lb}{\cancel{gal}}\right)$$

$$\Rightarrow \begin{cases} \dfrac{dQ}{dt} = 3 \cdot 2 - 3 \cdot \dfrac{Q}{300} \\ Q(0) = 50 \end{cases}$$

Solve this equation:

$$\begin{cases} Q' = 6 - \frac{Q}{100} = \frac{600 - Q}{100} \\ Q(0) = 50 \end{cases}$$

$$\frac{dQ}{dt} = -\frac{1}{100}(Q - 600)$$

$$\int \frac{dQ}{Q-600} = \int -\frac{1}{100}\,dt \Rightarrow \ln|Q-600| = -\frac{t}{100} + C$$

$$|Q - 600| = Ce^{-t/100}$$

$$Q = 600 + Ce^{-t/100}$$

$$Q(0) = 600 + C = 50 \Rightarrow C = -550$$

$$\Rightarrow Q(t) = 600 - 550e^{-t/100} \xrightarrow[t\to\infty]{} \underline{\underline{600}} \text{ (lbs)}.$$

How much salt will be in the tank after a long time.

$$c(t) = \frac{Q(t)}{V(t)} = \frac{600}{300} = 2 \left(\frac{lb}{gal}\right)$$

<u>Question</u>: ① if in this problem $r_{in} = 3$ gal/min, $r_{out} = 4$ gal/min

$$V(t) = 300 + (r_{in} - r_{out})t = 300 - t$$

When $t = 300$ (mins), $V(t) = 0 \to$ tank empties.

$$\Rightarrow Q(300) = 0.$$

② if in this problem $r_{in} = 3$ gal/min, $r_{out} = 2$ gal/min

if $V_{max} = 500$ gal (capacity)

When will the tank overflow

$$V(t) = 300 + (r_{in} - r_{out})t = 300 + t$$

$V(200) = 500 \Rightarrow$ tank overflows after 200 mins.