Math 214.
Lecture 3

Classification of ODEs :

1) ODEs vs. PDEs
   ↑ ordinary ↑ partial

   $\frac{dy}{dt}$ $\frac{\partial y}{\partial t}, \frac{\partial y}{\partial x}$ etc.

2) systems of DE: $\begin{cases} x' = f(x, y, t) \\ y' = g(x, y, t) \end{cases}$

   vs. single DE $x' = f(x, t)$ ← explicit

   or $f(x, x', t) = 0$ ← implicit

3) Order of DE = order of highest derivative involved

   ex. $y''' + 2y' = 0 \qquad y = y(t)$

   $t^5 y''' + 2y' = 0$

   $t y'' + (t^6 + 2t^4) y + \cos t = 0$

   $t \cdot y \cdot y'' + 2y' = 0$

4) Linear DE: $b(t) = a_n(t) y^{(n)} + a_{n-1}(t) y^{(n-1)} + \ldots + a_1(t) y' + a_0(t) y$

   others are nonlinear

   Ex. $t y'' + (t^6 + 2t^4) y + \cos t = 0$

   $a_2(t) y'' + a_1(t) y' + a_0(t) y = b(t)$

   $a_2(t) = t \quad a_1(t) = 0 \quad a_0(t) = t^6 + 2t^4$

   $b(t) = -\cos t$

Ex. $t^2 y''' + \frac{\cos t}{t} \cdot y' = t^4 + \tan t$ linear 3rd order

$t(y'')^2 + (t^2+2)y = 2$ ← nonlinear 2nd order

$y \cdot y' + 2ty + \frac{1}{\tan t} = 0$ ← nonlinear 1st order

$y^{(4)} + (\tan t)^{10} \cdot y' - \frac{y''}{t^2+1} = t^{100} \cdot y$ ← linear 4th order

§2.1. Integrating factors.

$$\boxed{y' + p(t)y = g(t)}$$

↑ Standard form of 1st order linear ODE.

$a(t)y' + b(t)y = f(t)$

$\Rightarrow y' + \underbrace{\frac{b(t)}{a(t)}}_{p(t)} y = \underbrace{\frac{f(t)}{a(t)}}_{g(t)}$ if $a(t) \neq 0$.

Ex. $y' + 2y = 3$

You know how to separate variables:

Method 1: $\frac{dy}{dt} = -2y + 3$

$\frac{dy}{dt} = -2(y - \frac{3}{2})$

$\int \frac{dy}{y - \frac{3}{2}} = -\int 2\,dt \Rightarrow \ln|y - \frac{3}{2}| = -2t + C$

$$\boxed{y = \frac{3}{2} + Ce^{-2t}}$$

Method 2: $y' + 2y = 3 \quad | * e^{2t}$ $\qquad \mu = e^{2t}$ integrating factor

$y' e^{2t} + 2e^{2t} y = 3e^{2t}$

$\frac{d}{dt}(y \cdot e^{2t}) = y' \cdot e^{2t} + y \cdot (2e^{2t})$

$\Rightarrow \int (y \cdot e^{2t})' = \int 3e^{2t}$

$\Rightarrow y \cdot e^{2t} = \frac{3}{2} e^{2t} + C$ / divide by e^{2t}

$$\boxed{y(t) = \frac{3}{2} + Ce^{-2t}}$$

In general if $\boxed{y' + ay = b} \Rightarrow \mu(t) = e^{at}$

$$e^{at}y' + ae^{at}y = (e^{at}y)'$$

$$(\mu(t) \cdot y)' = b\mu(t)$$

$$\mu(t)y(t) = \int b\mu(t)dt + C$$

$$e^{at}y(t) = \int be^{at}dt + C$$

$$e^{at}y(t) = \frac{b}{a}e^{at} + C$$

$$\Rightarrow \boxed{y(t) = \frac{b}{a} + Ce^{-at}}$$

Most general case:

$$y' + p(t)y = g(t) \qquad \boxed{\mu = e^{\int p(t)dt}}$$

$$\mu(t)y' + \mu(t) * p(t)y = g(t) \cdot \mu(t)$$

We want: $(\mu(t) \cdot y)' = \mu(t) \cdot y' + \mu'(t) \cdot y$

Show:

$$\mu'(t)y = \mu(t)p(t)y$$

$$\mu'(t) = \mu(t)p(t)$$

$$\int \frac{d\mu(t)}{\mu(t)} = \int p(t)\,dt$$

$$\ln|\mu(t)| = \int p(t)dt + C$$

$$\boxed{\mu(t) = C \cdot e^{\int p(t)dt}}$$

Choose $C = 1$ so that integrating factor becomes $\mu(t) = e^{\int p(t)dt}$

We get: $(\mu(t) \cdot y)' = g(t) \cdot \mu(t)$

Integrate: $\mu(t) \cdot y(t) = \int g(s)\mu(s)ds + C$

$$y(t) = e^{-\int p(t)dt} \cdot \left[\int g(s)\mu(s)ds + C \right]$$

Outline of the method:

1) Put ODE in standard form

$$\boxed{y' + p(t)y = g(t)} \quad (*)$$

2) Identify $p(t)$ and find integrating factor $\mu(t) = e^{\int p(t)\,dt}$

3) Multiply both sides by $\mu(t)$ to get $(\mu \cdot y)' = \mu(t) g(t)$

4) Integrate both sides and solve for $y(t)$ which will give general solution of $(*)$.

Ex. 1. $y' + \underbrace{\frac{1}{t}}_{p(t)} y = \underbrace{e^t}_{g(t)}$ (if $ty' + y = te^t$)

$p(t) = \frac{1}{t}$

$$\mu(t) = e^{\int p(t)\,dt} = e^{\int \frac{1}{t}dt + \bar{C}} = Ce^{\ln t} = Ct$$

Put $C=1$ since we choose one solution of this family

$$ty' + y = te^t$$

$$(\mu(t)\cdot y)' = (t \cdot y)' = te^t \Rightarrow \text{Integrate } \int (ty)' = \int se^s + C$$

$$t\cdot y = \int_0^t se^s\,ds + C = te^t - e^t + C$$

$$\boxed{\int u\,dv = uv - \int v\,du} \leftarrow \text{Integration by parts}$$

$$\int_0^t se^s\,ds = te^t - \int e^s\,ds = te^t - e^t$$

$u = s \quad dv = e^s$

$du = ds \quad v = e^s$

$$\boxed{y(t) = e^t - \frac{e^t}{t} + \frac{C}{t}}, \quad t \neq 0$$

Ex. 2 $\quad ty' + 2y = t^2$

$$\boxed{e^{a \ln t} = t^a}$$