Math 214.

Lecture 22.

Linear systems: $\vec{x}' = A\vec{x} + g$

Required: $\vec{x}' = A\vec{x}$ (*)

const coeff linear homogeneous system of ODEs.

We know: if $\vec{x}^{(1)}, \vec{x}^{(2)}$ - solutions to (*) and $W(\vec{x}^{(1)}, \vec{x}^{(2)}) \neq 0$ then $\vec{x}^{(1)}, \vec{x}^{(2)}$ - form a fundam. set, i.e.

$\vec{x}(t) = C_1 \vec{x}^{(1)}(t) + C_2 \vec{x}^{(2)}(t)$ is the general solution to (*).

Suppose $\vec{x}' = A\vec{x}$ is your system

Take $x = \xi e^{rt}$ — one solution of $x' = Ax$

const vector

Plug into $x' = Ax$: $\begin{cases} x' = \xi \cdot r e^{rt} \\ Ax = A(\xi e^{rt}) = e^{rt} \cdot A\xi \end{cases} \Rightarrow r\xi e^{rt} = A\xi e^{rt}$

$\boxed{r\xi = A\xi}$

So for $x = \xi e^{rt}$ to be a solution to (*), it has to satisfy $A\xi = r\xi$.

r - eigenvalue

ξ - eigenvector of A.

Case 1. Suppose A is a 2×2 matrix with eigenvalues r_1, r_2

ξ_1, ξ_2 - corresp. eigenvectors

$\boxed{r_1 \neq r_2, \text{ real}}$

Then $\left.\begin{array}{l} x^{(1)} = \xi_1 e^{r_1 t} \\ x^{(2)} = \xi_2 e^{r_2 t} \end{array}\right\} \Rightarrow W(x^{(1)}, x^{(2)}) \neq 0$

So in this case

Gen. sol. of $x' = Ax$: $\boxed{x(t) = C_1 \xi_1 e^{r_1 t} + C_2 \xi_2 e^{r_2 t}}$

Ex. $x' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x$ $\quad A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$

Find gen. solution.

1) Find eig. values: $\det(A - rI) = 0$

$$\begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = (1-r)(1-r) - 4 = 1 - 2r + r^2 - 4$$

$$= r^2 - 2r - 3 = 0$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad (r-3)(r+1) = 0$$

$$\underbrace{r_1 = 3 \quad r_2 = -1}_{\text{eig. values}}$$

2) Find eig. vectors

Solve system: $(A - rI)\vec{\xi} = 0$

eig. vector corresp to r

for $r_1 = 3$: $\begin{pmatrix} 1-3 & 1 \\ 4 & 1-3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} -2v_1 + v_2 = 0 \\ 4v_1 - 2v_2 = 0 \end{matrix}$

$\vec{\xi}_1 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $\qquad \boxed{v_2 = 2v_1}$

$\boxed{\xi_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$ $\leftarrow$ $v_1 = 1 \Rightarrow v_2 = 2$

for $r_2 = -1$: $\begin{pmatrix} 1-(-1) & 1 \\ 4 & 1-(-1) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \begin{matrix} 2v_1 + v_2 = 0 \\ 4v_1 + 2v_2 = 0 \end{matrix}$

$\boxed{\xi_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$ $\Leftarrow$ $\boxed{v_2 = -2v_1}$

Gen. solution: $\boxed{x(t) = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + C_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}}$

For phase portrait:

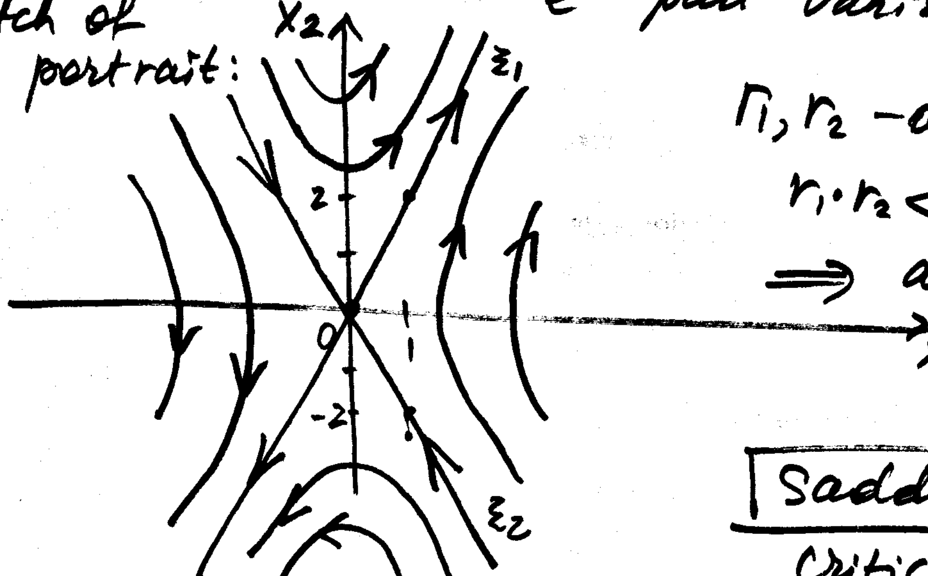
$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} \begin{pmatrix}1\\2\end{pmatrix} e^{3t} \longrightarrow x(t) \text{ is parallel to } \begin{pmatrix}1\\2\end{pmatrix}$$

e^{-t} part vanishes

At $t\to-\infty \Rightarrow$ $x(t)$ will be parallel to $\begin{pmatrix}1\\-2\end{pmatrix}$

e^{3t} part vanishes

Sketch of phase portrait:



r_1, r_2 – different sign

$r_1 \cdot r_2 < 0$

$\Longrightarrow$ always produce similar picture

$\boxed{\text{Saddle}}$ $\boxed{\text{unstable}}$

Critical point at $(0,0)$.

Ex. 2

$$x' = \begin{pmatrix}1 & -2\\ 3 & -4\end{pmatrix} x$$

$$\begin{vmatrix}1-r & -2\\ 3 & -4-r\end{vmatrix} = -4 - r + 4r + r^2 + 6 = 0$$

$$r^2 + 3r + 2 = 0$$

$$(r+2)(r+1) = 0 \quad r_1 = -2 \quad r_2 = -1$$

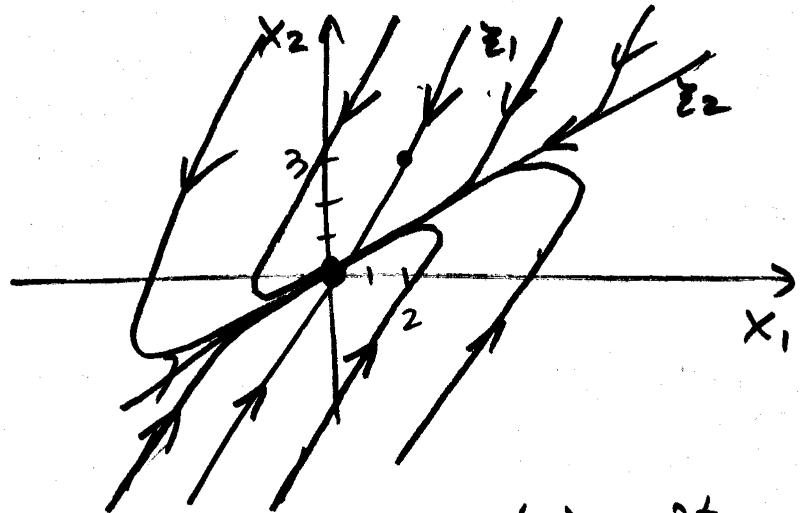
$$\boxed{\begin{array}{c}\text{Stable node}\\ r_1, r_2 < 0 \text{ real } r_1 \neq r_2\end{array}}$$

For ξ_1: $\begin{pmatrix}3 & -2\\ 3 & -2\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$3v_1 - 2v_2 = 0 \qquad \xi_1 = \begin{pmatrix}2\\3\end{pmatrix}$

$v_1 = \frac{2}{3} v_2$

ξ_2: $\begin{pmatrix}2 & -2\\ 3 & -3\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{array}{c}2v_1 - 2v_2 = 0\\ v_1 = v_2\end{array} \qquad \xi_2 = \begin{pmatrix}1\\1\end{pmatrix}$



stable node

$r_1, r_2 < 0$ $r_1 \neq r_2$ real.

Gen. sol. : $x(t) = C_1\begin{pmatrix}2\\3\end{pmatrix}e^{-2t} + C_2\begin{pmatrix}1\\1\end{pmatrix}e^{-t}$

At $t \to \infty$ $x(t) \parallel \begin{pmatrix}1\\1\end{pmatrix}$ since e^{-2t} is dominated by e^{-t}, $t>0$.

At $t \to -\infty$ $x(t) \parallel \begin{pmatrix}2\\3\end{pmatrix}$ Since e^{-2t} dominates e^{-t} at $t<0$.

Classification of critical pts of $x' = Ax$.

Case 1. $r_1 \neq r_2$, real

 (a) $r_1, r_2 > 0 \Rightarrow$ unstable node at $(0,0)$.

all trajectories are repelled from 0.

 (b) $r_1, r_2 < 0 \Rightarrow$ asymptotically stable node at $(0,0)$

all traj. are attracted to 0.

 (c) $r_1 \cdot r_2 < 0$ (different signs) $\Rightarrow$ unstable saddle.

the traj. are first attracted then repelled unless it coincides with an eigenvector (called separatrix).

Case 2. r_1, r_2 - complex.

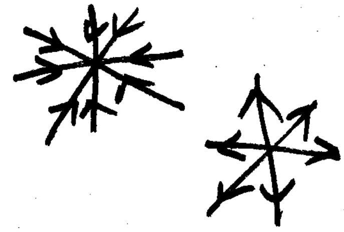
$r_{1,2} = \lambda \pm i\mu \Rightarrow$ Spiral

$\lambda > 0 \Rightarrow$ unstable spiral , $\lambda < 0$ stable spiral 

Case 3. $r_1 = r_2$ real degenerate node

$r_1, r_2 < 0 \Rightarrow$ stable

$r_1, r_2 > 0 \Rightarrow$ unstable.



Ex. $x' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x$ Find gen. sol. and sketch phase portrait

$r_1 = 1 \quad r_2 = -1$

$\xi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \xi_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$