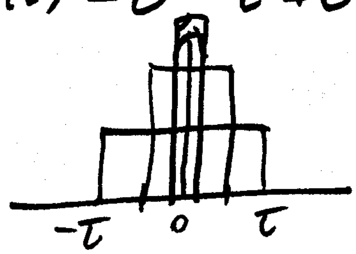
Math 214.
Lecture 21.

§6.5. Impulse functions.

$$\underbrace{I(\tau) = \int_{-\infty}^{+\infty} g(t)\,dt}_{\text{total impulse of the force } g(t) \text{ over } (t_0-\tau,\, t_0+\tau)}, \quad g(t) = \begin{cases} \text{large} & \text{in } t_0-\tau<t<t_0+\tau \\ 0, & \text{otherwise} \end{cases}$$

Define $d_\tau(t) = \begin{cases} \frac{1}{2\tau}, & -\tau<t<\tau \\ 0, & \text{otherwise} \end{cases}$

Let $\tau \to 0 \Rightarrow \lim_{\tau\to 0} d_\tau(t) = 0 \quad t \neq 0.$



$\tau = 1 \Rightarrow \frac{1}{2\tau} = \frac{1}{2}$

$\tau = \frac{1}{2} \Rightarrow \frac{1}{2\tau} = 1$

$\tau = \frac{1}{4} \Rightarrow \frac{1}{2\tau} = 2$

$$I(\tau) = \int_{-\infty}^{+\infty} d_\tau(t)\,dt = \int_{-\tau}^{\tau} \frac{1}{2\tau}\,dt = \frac{1}{2\tau}\cdot 2\tau = 1 \quad \text{indep. of } \tau$$

So $\lim_{\tau\to 0} I(\tau) = 1$

Define <u>unit impulse function</u> δ (Dirac function, generalized function)

$\delta(t) = 0,\ t \neq 0$ and $\int_{-\infty}^{+\infty} \delta(t)\,dt = 1.$

Can show: $\displaystyle\int_{-\infty}^{+\infty} \delta(t-t_0) f(t)\,dt = f(t_0)$

Laplace transform:

$$\left| \begin{array}{l} \mathcal{L}\{\delta(t-t_0)\} = e^{-st_0} \\ \mathcal{L}\{\delta(t)\} = \lim_{t_0\to 0} e^{-st_0} = 1 \end{array} \right|$$

<u>Ex. 1</u> $y'' + 2y' + 2y = \delta(t-\pi),\ y(0)=1,\ y'(0)=0$

Same way as before:

$\mathcal{L}\{y'' + 2y' + 2y\} = \mathcal{L}\{\delta(t-\pi)\}$

$$s^2F(s) - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{0} + 2(sF(s) - \underbrace{y(0)}_{1}) + 2F(s) = e^{-\pi s}$$

$$(s^2+2s+2)F(s) - s - 2 = e^{-\pi s}$$

$$F(s) = \frac{s+2}{s^2+2s+2} + \frac{e^{-\pi s}}{s^2+2s+2} =$$

$$(s+1)^2+1$$

$$= \boxed{\frac{s+2}{(s+1)^2+1}} + \frac{e^{-\pi s}}{(s+1)^2+1}$$

Line 9: $\frac{b}{(s-a)^2+b^2} \xrightarrow{\mathcal{L}^{-1}} e^{at}\sin bt$ $\qquad a=-1$, $b=1$

Line 10: $\frac{s-a}{(s-a)^2+b^2} \xrightarrow{\mathcal{L}^{-1}} e^{at}\cos bt$

$$\frac{s+2}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}$$

$$\downarrow \qquad\qquad \downarrow$$

$$e^{-t}\cos t \qquad e^{-t}\sin t$$

$$\frac{e^{-\pi s}}{(s+1)^2+1} = e^{-\pi s}\cdot H(s), \text{ where } H(s) = \frac{1}{(s+1)^2+1}$$

$$h(t) = e^{-t}\sin t$$

Line 13: $e^{-cs}F(s) \xrightarrow{\mathcal{L}^{-1}} u_c(t)\, f(t-c)$

$$y(t) = \boxed{e^{-t}\cos t + e^{-t}\sin t} + u_\pi(t)\, h(t-\pi)$$

$$= e^{-t}\cos t + e^{-t}\sin t + u_\pi(t)\cdot e^{-(t-\pi)}\underbrace{\sin(t-\pi)}_{=-\sin t}$$

Ex. 2 $y^{(4)} - y = \delta(t-1)$, $y(0)=0$, $y'(0)=0$, $y''(0)=0$, $y'''(0)=0$.

$$\mathcal{L}\{y^{(4)}\} = s^4F(s) - s^3\cancel{y(0)}^{0} - s^2\cancel{y'(0)}^{0} - s\cancel{y''(0)}^{0} - \cancel{y'''(0)}^{0} \text{ by Line 18}$$

$$s^4F(s) - F(s) = e^{-s}$$

$$F(s) = \frac{e^{-s}}{s^4-1} = e^{-s}\cdot H(s) \qquad \boxed{c=1}$$

$$H(s) = \frac{1}{s^4-1} \Rightarrow h(t) = \mathcal{L}^{-1}(H(s))$$

form of Answer: $y(t) = u_1(t)h(t-1)$ where

$$h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^4-1}\right)$$

$$\frac{1}{s^4-1} = \frac{1}{(s^2-1)(s^2+1)} = \frac{1}{(s-1)(s+1)(s^2+1)}$$

$$\frac{1}{(s-1)(s+1)(s^2+1)} = \frac{a}{s-1} + \frac{b}{s+1} + \frac{cs+d}{s^2+1}$$

$$a(s^3+s+s^2+1) + b(s^3+s-s^2-1) + (cs+d)(s^2-1) = 1$$

$$cs^3 - cs + ds^2 - d$$

$$\begin{cases} a+b+c=0 \\ a-b+d=0 \\ a+b-c=0 \\ a-b-d=1 \end{cases} \quad + \quad +$$

$2(a+b)=0$

$a+b=0$ $\quad a=-b$

$2(a-b)=1$

$a-b=\frac{1}{2}$ $\quad 2a=\frac{1}{2}$ $\quad a=\frac{1}{4}$, $b=-\frac{1}{4}$

$c = a+b = 0$

$d = b-a = -\frac{1}{2}$ $\quad \Rightarrow \frac{1}{s^4-1} = \frac{\frac{1}{4}}{s-1} - \frac{\frac{1}{4}}{s+1} - \frac{\frac{1}{2}}{s^2+1}$

$$\Rightarrow \boxed{\begin{array}{l} h(t) = \frac{1}{4}e^{t} - \frac{1}{4}e^{-t} - \frac{1}{2}\sin t \\ y(t) = u_1(t)h(t-1) \text{ where} \end{array}}$$

§7. Systems of 1st order Linear ODEs:

$$\begin{cases} x_1' = F_1(x_1, \dots x_n, t) \\ \vdots \\ x_n' = F_n(x_1, \dots x_n, t) \end{cases}$$

in vector form $\vec{x}' = \vec{F}(\vec{x}, t)$

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \vec{F} = \begin{pmatrix} F_1(\vec{x}, t) \\ \vdots \\ F_n(\vec{x}, t) \end{pmatrix}$$

Fact:

Any n-th order linear ODE of the form $y^{(n)} = F(t, y, y', \dots y^{(n-1)})$ is equivalent to a system of 1st order linear ODEs.

How it works: change variables

$$\begin{cases} x_1 = y \\ x_2 = y' \\ \vdots \\ x_{n-1} = y^{(n-2)} \\ x_n = y^{(n-1)} \end{cases} \Longrightarrow \begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \vdots \\ x_{n-1}' = x_n \\ x_n' = y^{(n)} = F(t, x_1, x_2, \dots x_n) \end{cases}$$

System of linear 1st order ODEs.

Ex. $y'' + 2y' + 3y = 0$

Transform this eqn into a system of 1st order ODEs:

$$\left.\begin{matrix} x_1 = y \\ x_2 = y' \end{matrix}\right\} \Rightarrow \begin{cases} x_1' = \underline{x_2} \\ x_2' = y'' = -2y' - 3y = \underline{-3x_1 - 2x_2} \end{cases}$$

System is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} x_2 \\ -3x_1 - 2x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\begin{pmatrix} \boxed{0} x_1 + \boxed{1} \cdot x_2 \\ \boxed{-3} x_1 \boxed{-2} \cdot x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\vec{x}' = P \cdot \vec{x}$ where $P = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix}$

All linear systems in matrix form can be written as $\vec{x}' = P(t)\vec{x} + \vec{g}(t)$

For the homogeneous eqn $\vec{x}' = P(t)\vec{x}$ (*) :

Thm 1. $x^{(1)}, x^{(2)}$ - solns of (*) $\Rightarrow$ $c_1 x^{(1)} + c_2 x^{(2)}$ is also a solution of (*)

Thm 2. $x^{(1)}, \dots x^{(n)}$ - lin. indep. solutions of (*) $\Rightarrow$ $\varphi(t) = c_1 x^{(1)} + \dots + c_n x^{(n)}$ for any $c_1, \dots c_n$ is the gen. solution of (*).

Thm 3. If Wronskian is computed as $W[x^{(1)}, \dots x^{(n)}] = \det[x^{(1)} \dots x^{(n)}]$ then $W[x^{(1)}, \dots x^{(n)}] \equiv 0$ on $[\alpha, \beta]$ or never vanishes on $[\alpha, \beta]$ for any $x^{(1)}, \dots x^{(n)}$ - solns of (*) at $[\alpha, \beta]$.

Example.

If $x^{(1)} = \begin{pmatrix} t \\ 1 \end{pmatrix}$, $x^{(2)} = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$ - solutions to same system $x' = Px$

$$W(x^{(1)}, x^{(2)}) \underset{\det}{=} \begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix} = 2t^2 - t^2 = t^2 \neq 0$$ for any interval $[\alpha, \beta]$ not containing 0.