Math 214.
Lecture 2.

Separation of variables:

$\frac{dy}{dt} = ay + b$, a, b – constants
$y(t)$ – sought solution

$$\frac{dy}{dt} = a\left(y + \frac{b}{a}\right)$$

$$\int \frac{dy}{y + \frac{b}{a}} = \int a\,dt$$

$$\ln\left|y + \frac{b}{a}\right| = at + C$$

$$\left|y + \frac{b}{a}\right| = e^{at+C} = e^{C} \cdot e^{at}$$

$$y + \frac{b}{a} = \underbrace{\pm e^{C}}_{\text{call this } C \text{ again}} \cdot e^{at} = Ce^{at}$$

$$\Rightarrow \boxed{y(t) = -\frac{b}{a} + Ce^{at}}$$ – general solution to $y' = ay + b$

If $y(0) = y_0$ then $\underset{\overset{\parallel}{y_0}}{y(0)} = -\frac{b}{a} + C$
$\Rightarrow C = y_0 + \frac{b}{a}$

Initial value problem $\begin{cases} \frac{dy}{dt} = ay + b \\ y(0) = y_0 \end{cases}$

has the solution of the form $\boxed{y(t) = -\frac{b}{a} + \left(y_0 + \frac{b}{a}\right)e^{at}}$

solution to IVP

Ex. $\begin{cases} \frac{dx}{dt} = 5x+2 \\ x(0) = x_0 \end{cases}$

$\frac{dx}{dt} = 5(x+\frac{2}{5})$

$\int \frac{dx}{x+\frac{2}{5}} = \int 5\,dt \Rightarrow \ln|x+\frac{2}{5}| = 5t + C$

$x + \frac{2}{5} = Ce^{5t}$

$x = -\frac{2}{5} + Ce^{5t}$

$x(0) = -\frac{2}{5} + C = x_0 \Rightarrow C = x_0 + \frac{2}{5}$

$$\boxed{x(t) = -\frac{2}{5} + \left(x_0 + \frac{2}{5}\right)e^{5t}}$$

Ex. $\begin{cases} \frac{dx}{dt} = 2 - x \\ x(0) = 1 \end{cases}$ Solve this IVP

Ex.2 $\begin{cases} \frac{dx}{dt} = 3x+1 \\ x(0) = x_0 \end{cases}$

For which x_0 is the solution bounded?

Checking that a function is a solution:

Ex. $x = t$ ← verify this gives a solution to $x'' - tx' + x = 0$

$x' = 1 \quad x'' = 0 \Rightarrow 0 - t\cdot 1 + t = 0$ ✓

$-t + t = 0$

Ex. 2 Which of the following solve $x''' = 0$?

(a) $x = t^3$, (b) $x = at^2 + bt + c$

$x' = 3t^2$ $\quad$ $x' = 2at + b$

$x'' = 6t$ $\quad$ $x'' = 2a$

$x''' = 6$ $\quad$ $x''' = 0$

Problem:

Mice eaten by owls: $\frac{dp}{dt} = 0.5p - 450$

$p(t)$ = population of mice

t = time (in months)

owls eat 15 mice/day

1) Determine Text when population becomes extinct if $p(0) = 850$. $(p(T_{ext}) = 0)$

2) Find Text if $p(0) = p_0$, $0 < p_0 < 900$

3) Find $p_0 = p(0)$ if Text = 1 year

Solution:

1) $\frac{dp}{dt} = 0.5(p - 900)$

$\int \frac{dp}{p-900} = \int 0.5\, dt \Rightarrow \ln|p - 900| = \frac{1}{2}t + C$

$p - 900 = Ce^{t/2}$

gen. sol. $\boxed{p = 900 + Ce^{t/2}}$

$p(0) = 850 \Rightarrow p(0) = 900 + C = 850$

$C = -50$

$p(t) = 900 - 50e^{t/2}$

$0 = p(T_{ext}) = 900 - 50 \cdot e^{T_{ext}/2} \Rightarrow$

$$900 = 50e^{T_{ext}/2}$$

$$18 = e^{T_{ext}/2} \Rightarrow \ln 18 = \frac{T_{ext}}{2}$$

$$\Rightarrow \boxed{T_{ext} = 2\ln 18 \text{ (months)}}$$

2) if $\underline{p(0) = p_0}$, $0 < p_0 < 900$

$$\Rightarrow \quad p(0) = 900 + C = p_0 \Rightarrow p_0 - 900 = C$$

$$p(t) = 900 + (p_0 - 900)e^{t/2}$$

$$0 = p(T_{ext}) = 900 + (p_0 - 900)e^{t/2}$$

$$\Rightarrow \quad 900 = (900 - p_0)e^{t/2}$$

$$\frac{900}{900 - p_0} = e^{t/2}$$

$$T_{ext} = 2\ln\left(\frac{900}{900 - p_0}\right)$$

3) if $T_{ext} = 12$ (months) find $p(0)$