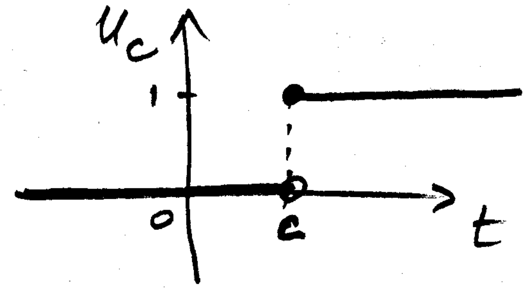
Math 214.
Lecture 19.

§6.3 Step functions.

Def $u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$ $\quad c \geq 0$



Ex. 1) $u_3(t) = \begin{cases} 0, & t<3 \\ 1, & t \geq 3 \end{cases}$

2) $u_3(t) - u_5(t) = \begin{cases} 0 & , t<3 \\ 1-0, & 3 \leq t < 5 \\ 1-1 & t \geq 5 \end{cases} = \begin{cases} 0, & t<3 \\ 1, & 3 \leq t < 5 \\ 0, & t \geq 5 \end{cases}$

$u_3 = 0 \quad u_3 = 1 \quad u_3 = 1$

$u_5 = 0 \quad 3 \quad u_5 = 0 \quad 5 \quad u_5 = 1$

3) $u_1(t) + 2u_5(t) - tu_6(t) = \begin{cases} 0, & t<1 \\ 1, & 1 \leq t < 5 \\ 3, & 5 \leq t < 6 \\ 3-t, & t \geq 6 \end{cases}$

$u_1 = 0 \quad u_1 = 1 \quad u_1 = 1 \quad u_1 = 1$

$u_5 = 0 \quad 1 \quad u_5 = 0 \quad 5 \quad u_5 = 1 \quad 6 \quad u_5 = 1$

$u_6 = 0 \quad u_6 = 0 \quad u_6 = 0 \quad u_6 = 1$

$0+0+0 \quad 1+0+0 \quad 1+2+0$

4) Find $f(5)$ if $f(t) = 2u_2(t) + (5-t^2)u_4(t) + u_5(t)$

$f(5) = 2 \cdot 1 + (5-5^2) \cdot 1 + 1 = 2 - 20 + 1 = \underline{\underline{-17}}$

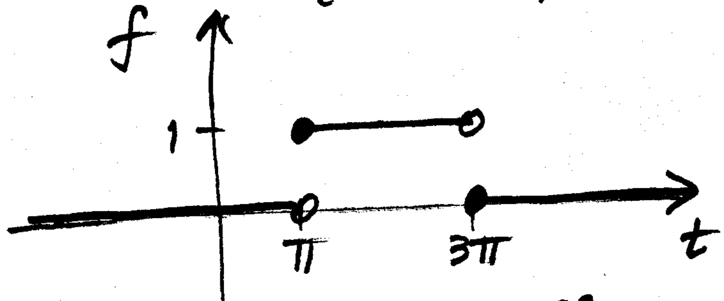
5) Find $f(3)$ if $f(t) = (2-t)\underset{1}{u_1(t)} + t^2 \underset{0}{u_4(t)}$

$f(3) = (2-3) \cdot 1 + 3^2 \cdot 0 = \underline{\underline{-1}}$

6) $f(t) = u_\pi(t) - u_{3\pi}(t)$, $t \geq 0$

Sketch $f(t)$.

$$f(t) = \begin{cases} 0 & , t < \pi \\ 1-0 & , \pi \leq t < 3\pi \\ 1-1 & , t \geq 3\pi \end{cases} = \begin{cases} 0 & , t < \pi \\ 1 & , \pi \leq t < 3\pi \\ 0 & , t \geq 3\pi \end{cases}$$



Fact: $\mathcal{L}\{u_c(t)\} = \dfrac{e^{-cs}}{s}$, $s > 0$

Laplace of $u_c(t)$

Two Translation Theorems.

Thm 1: $\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$

$u_c(t) f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}$, $F(s) = \mathcal{L}\{f\}$.

Thm 2. $\mathcal{L}\{e^{ct} f(t)\} = F(s-c)$, $s > a+c$, $F(s) = \mathcal{L}\{f\}$.

$e^{ct} f(t) = \mathcal{L}^{-1}\{F(s-c)\}$

Examples:

$g(t) = \begin{cases} 0, & t < 2 \\ (t-2)^2, & t \geq 2 \end{cases}$ Find $\mathcal{L}\{f\}$.

Step 1: Represent $f(t)$ by means of step functions.

$$g(t) = (\ 0\ ) + (t-2)^2 u_2(t) = (t-2)^2 u_2(t).$$

Step 2. Use Thm 1 to find $\mathcal{L}\{f\}$.

$$g(t) = (t-2)^2 u_2(t)$$

by Thm 1: $\underline{\underline{f(t-2)\,u_2(t)}} \xrightarrow{\mathcal{L}} e^{-2s}\mathcal{L}\{f\}$

if you have

Need to find $f(t)$ s.t.

$$f(t-2)u_2(t) = (t-2)^2 u_2(t)$$

$$f(\underline{t-2}) = (t-2)^2$$

$$f(t) = t^2$$

Hence $\mathcal{L}\{(t-2)^2 u_2(t)\} = e^{-2s}\mathcal{L}\{t^2\} = e^{-2s}\left(\frac{2}{s^3}\right)$

Ex. 2. let $g(t) = \begin{cases} 2, & t \geq 3 \\ t, & t < 3 \end{cases}$ Find $\mathcal{L}\{g\}$.

1) Write $g(t)$ in terms of step fcts.

$$g(t) = (\ t\ ) + (\ 2-t\ )u_3(t)$$

2) Take $\mathcal{L}\{g\} = \underbrace{\mathcal{L}\{t\}}_{\frac{1}{s^2}} + \underbrace{\mathcal{L}\{(2-t)u_3(t)\}}_{\text{by Thm 1}}$

$$(2-t)u_3(t) = f(t-3)u_3(t)$$

$$\left.\begin{aligned} 2-t &= f(\underline{t-3}) \\ &\quad \tau = t-3 \\ &\quad \tau+3 = t \\ 2-t &= 2-(\tau+3) = -1-\tau \end{aligned}\right\} \Rightarrow f(\tau) = -\tau-1$$

$$\mathcal{L}\{g\} = \mathcal{L}\{t\} + \mathcal{L}\{f(t-3)u_3(t)\} = \frac{1}{s^2} + e^{-3s}\mathcal{L}\{-t-1\}$$

$f(t) = -t-1$

$$= \frac{1}{s^2} + e^{-3s}\left(-\frac{1}{s^2} - \frac{1}{s}\right)$$

$$\boxed{\mathcal{L}\{g\} = \frac{1}{s^2}(1-e^{-3s}) - \frac{e^{-3s}}{s}}$$

Ex. 3 . $g(t) = \begin{cases} t, & t < 1 \\ e^t + e^{-t}, & 1 \le t < 2 \\ 1, & t \geq 2 \end{cases}$

Write $g(t)$ in terms of step functions.

$$g(t) = (\ t\ ) + (e^t + e^{-t} - t)u_1(t) + (1 - e^t - e^{-t})u_2(t)$$

$$(e^t + e^{-t} - t)u_1(t) = f(t-1)u_1(t)$$

$$e^t + e^{-t} - t = f(t-1)$$

$$\tau = t - 1 \Rightarrow t = \tau + 1$$

$$e^{\tau+1} + e^{-(\tau+1)} - (\tau+1) = f(\tau)$$

$$\mathcal{L}\{(e^t + e^{-t} - t)u_1(t)\} = e^{-s}\mathcal{L}\{e^{t+1} + e^{-(t+1)} - (t+1)\}$$

$$= e^{-s}\mathcal{L}\{e \cdot e^t + e^{-1} \cdot e^{-t} - t - 1\}$$

$$= e^{-s} \cdot \left[e \cdot \frac{1}{s-1} + e^{-1} \cdot \frac{1}{s+1} - \frac{1}{s^2} - \frac{1}{s}\right]$$