Math 214.
Lecture 18.

Def. $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$

Laplace transform of f.

Strategy of using Laplace transform (for IVPs):

1. Take Laplace transform of both sides (from column ① to ② in the Table)
2. Solve for $F(s) = \mathcal{L}\{y\}$ where $y(t)$ is the solution
3. Take inverse Laplace transform of $F(s)$ to get $y(t)$. (from column ② to ① in Table).

Ex. $y'' - y' - 6y = 2$, $y(0)=0$, $y'(0)=0$

$\mathcal{L}(y'' - y' - 6y) = \mathcal{L}(2)$

$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 6\mathcal{L}\{y\} = \mathcal{L}\{2\}$

Denote $F(s) = \mathcal{L}\{y\}$

By line 18, $\begin{bmatrix} \mathcal{L}\{y''\} = s^2F(s) - sy(0) - y'(0) \\ \mathcal{L}\{y'\} = sF(s) - y(0) \end{bmatrix}$

By line 1, $\mathcal{L}\{2\} = 2\mathcal{L}\{1\} = \frac{2}{s}$

$$s^2F(s) - s\underset{0}{\underset{\shortparallel}{y(0)}} - \underset{0}{\underset{\shortparallel}{y'(0)}} - sF(s) + \underset{0}{\underset{\shortparallel}{y(0)}} - 6F(s) = \frac{2}{s}$$

$$s^2F(s) - sF(s) - 6F(s) = \frac{2}{s} \Rightarrow (s^2 - s - 6)F(s) = \frac{2}{s}$$

$$F(s) = \frac{2}{s(s^2-s-6)}$$

$$s^2 - s - 6 = (s-3)(s+2)$$

$$F(s) = \frac{2}{s(s+2)(s-3)} \leftarrow$$ Laplace transform of solution $y(t)$.

$$\frac{2}{s(s+2)(s-3)} = \frac{a}{s} + \frac{b}{s+2} + \frac{c}{s-3}$$ Partial fractions.

$$a(s+2)(s-3) + bs(s-3) + cs(s+2) = 2$$
$$a(s^2 - s - 6) + b(s^2 - 3s) + c(s^2 + 2s) = 2$$
$$(a+b+c)s^2 + (-a-3b+2c)s - 6a = 2$$

$$\begin{cases} a+b+c = 0 \\ -a-3b+2c = 0 \\ -6a = 2 \end{cases} \qquad 3\times\begin{cases} b+c = \frac{1}{3} \\ -3b+2c = -\frac{1}{3} \end{cases} \qquad a = -\frac{1}{3}$$

$$5c = \frac{2}{3} \quad \Leftarrow \begin{cases} 3b+3c = 1 \\ -3b+2c = -\frac{1}{3} \end{cases}$$

$$c = \frac{2}{15}$$

$$b = \frac{1}{3} - c = \frac{1}{3} - \frac{2}{15} = \frac{3}{15} = \frac{1}{5}$$

$$F(s) = \left(-\frac{1}{3}\right)\frac{1}{s} + \left(\frac{1}{5}\right)\frac{1}{s+2} + \left(\frac{2}{15}\right)\cdot\frac{1}{s-3}$$

Line 1: $\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$

Line 2: $\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) = e^{-2t}$

$\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) = e^{3t}$

$$y(t) = \mathcal{L}^{-1}(F(s)) = \left(-\frac{1}{3}\right)1 + \left(\frac{1}{5}\right)\cdot e^{-2t} + \left(\frac{2}{15}\right)\cdot e^{3t}$$

Let us practice taking $\mathcal{L}^{-1}$.

Ex. 1. $\mathcal{L}^{-1}\left\{\frac{4}{s+5}\right\} = 4\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} = 4\cdot e^{-5t}$

$a = -5$

Ex. 2 $\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t$

$a=1$ line 6

Ex. 3 $\mathcal{L}^{-1}\left\{\frac{s-3}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$

line 6 $a=2$ $\qquad$ $a=2$ line 5

$= \cos 2t - \frac{3}{2}\sin 2t$

Ex. 4 $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = \frac{1}{2}\cdot t^2$

$n=2$ $\quad \frac{n!}{s^{n+1}} = \frac{2}{s^3}$ line 3

Ex. 5. $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} = e^{-t}\cos t$

Cannot factor $\Rightarrow$ complete the square

$s^2+2s+2 = (s^2+2s+1)+1 = (s+1)^2+1$

in line 10 $a=-1$ $b=1$

$\Rightarrow \frac{s+1}{(s+1)^2+1}$

Ex. Obtain $F(s)$ for (Laplace of $y(t)$):

$y''+y = \sin 2t$, $y(0)=2$, $y'(0)=1$

$s^2F(s) - s\underbrace{y(0)}_{2} - \underbrace{y'(0)}_{1} + F(s) = \frac{2}{s^2+4}$

$(s^2+1)F(s) = 2s+1+\frac{2}{s^2+4}$

$F(s) = \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)}$

Partial fractions:

$$\frac{2}{(s^2+1)(s^2+4)} = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4}$$

$$(as+b)(s^2+4) + (cs+d)(s^2+1) = 2$$

$$\underline{as^3} + 4as + \underline{bs^2} + 4b + \underline{cs^3} + cs + \underline{ds^2} + d = 2$$

$$\begin{cases} a+c=0 \\ b+d=0 \\ 4a+c=0 \\ 4b+d=2 \end{cases} \qquad \begin{cases} a+c=0 \\ 4a+c=0 \end{cases} \qquad \begin{cases} b+d=0 \\ 4b+d=2 \end{cases}$$

$$3a=0 \qquad a=0 \quad c=0 \qquad\qquad 3b=2 \quad b=\tfrac{2}{3} \qquad d=-\tfrac{2}{3}$$

$$F(s) = \frac{2s+1}{s^2+1} + \frac{2}{3}\cdot\frac{1}{s^2+1} - \frac{2}{3}\cdot\frac{1}{s^2+4}$$

$$= 2\cdot\frac{s}{s^2+1} + \frac{1}{s^2+1} + \frac{2}{3}\cdot\frac{1}{s^2+1} - \frac{2}{3}\cdot\frac{1}{s^2+4}$$

$$= 2\cdot\frac{s}{s^2+1} + \frac{5}{3}\cdot\frac{1}{s^2+1} - \frac{2}{3}\cdot\frac{1}{s^2+4}$$

$$y(t) = \mathcal{L}^{-1}(F(s)) = 2\cos t + \frac{5}{3}\cdot\sin t - \frac{1}{3}\cdot\sin 2t$$

line 6 $a=1$ line 5 $a=1$ line 5 $a=2$

<u>Ex</u>. $y'' + 4y = e^t$, $y(0)=0$, $y'(0)=0$.

$$s^2F(s) - \cancel{sy(0)} - \cancel{y'(0)} + 4F(s) = \frac{1}{s-1}$$

$$(s^2+4)F(s) = \frac{1}{s-1} \Rightarrow F(s) = \frac{1}{(s-1)(s^2+4)}$$

$$\frac{1}{(s-1)(s^2+4)} = \frac{a}{s-1} + \frac{bs+c}{s^2+4} \qquad a=\tfrac{1}{5} \quad b=-\tfrac{1}{5} \quad c=-\tfrac{1}{5}$$

$$F(s) = \frac{1}{5}\cdot\frac{1}{s-1} - \frac{1}{5}\cdot\frac{s}{s^2+4} - \frac{1}{5}\,\frac{1}{s^2+4}$$

$$\left[y(t) = \frac{1}{5}e^t - \frac{1}{5}\cos 2t - \frac{1}{10}\sin 2t\right].$$