Math 214.

lecture 17

Vibrations: $mu'' + \gamma u' + ku = F(t)$

Special cases:

1) $\gamma = 0$ undamped , $\gamma \neq 0$ damped
2) $F = 0$ free , $F \neq 0$ forced

Things to remember:

1) Condition for getting k : $\boxed{mg = kL}$
   and L (elongation due to gravity)

2) $g = 32 \left(\frac{ft}{sec^2}\right)$, $1\ ft = 12\ in$
   English units: ft, lb, sec
   Metric units: m, kg, sec

3) If your answer is expressed as
   $u = R\cos(\omega t - \delta)$
   amplitude — frequency — phase shift , $T = \frac{2\pi}{\omega}$ - period
   Equivalent to $u = C_1 \cos\omega t + C_2 \sin\omega t$
   where $R = \sqrt{C_1^2 + C_2^2}$, $\tan\delta = \frac{C_2}{C_1}$

4) In the damped case,
   $u = Re^{-\gamma t/2m} \cos(\mu t - \delta)$
   quasifrequency

   $T = \frac{2\pi}{\mu}$ - quasiperiod

Ex. A mass of 1 kg is attached to a spring with $k = 3\left(\frac{N}{m}\right)$. A mass is acted upon by an external force of $2\sin t\,(N)$, and moves in a viscous fluid with resistance force $2u'\,(N)$. Write the DE and find the amplitude of the steady-state solution.

$$\left.\begin{array}{l} m = 1\ (kg) \\ k = 3\ \left(\frac{N}{m}\right). \\ F = 2\sin t\ (N). \\ \gamma = 2\ (\text{since } F_d = 2u') \end{array}\right\}$$

$$mu'' + \gamma u' + ku = F(t)$$

$$\boxed{u'' + 2u' + 3u = 2\sin t}$$

$$r^2 + 2r + 3 = 0.$$

$$r = \frac{-2 \pm \sqrt{4-12}}{2} = \frac{-2 \pm i2\sqrt{2}}{2} = -1 \pm i\sqrt{2}$$

$$\left[y_{gen.hom.} = C_1 e^{-t}\sin\sqrt{2}t + C_2 e^{-t}\cos\sqrt{2}t\right]$$

$y_p = A\sin t + B\cos t \Rightarrow$ plug in

$$y_p' = A\cos t - B\sin t$$

$$y_p'' = -A\sin t - B\cos t$$

$$\underline{-A\sin t} - \underline{B\cos t} + \underline{2A\cos t} - \underline{2B\sin t} + \underline{3A\sin t} + \underline{3B\cos t} = 2\sin t$$

$$(-A - 2B + 3A - 2)\sin t + (-B + 2A + 3B)\cos t = 0$$

$$\begin{cases} -A - 2B + 3A - 2 = 0 \\ -B + 2A + 3B = 0 \end{cases} \Rightarrow \begin{cases} 2A - 2B = 2 \\ 2A + 2B = 0 \end{cases}$$

$$\begin{cases} A - B = 1 \\ A + B = 0 \end{cases} \Rightarrow \quad 2A = 1 \quad A = \tfrac{1}{2} \quad B = -\tfrac{1}{2}$$

$$y_p = \tfrac{1}{2}\sin t - \tfrac{1}{2}\cos t$$

$$y_{gen.nonhom.} = \underbrace{C_1 e^{-t}\sin\sqrt{2}t + C_2 e^{-t}\cos\sqrt{2}t}_{\text{transient sol.}} + \underbrace{\tfrac{1}{2}\sin t - \tfrac{1}{2}\cos t}_{\text{Steady state sol.}}$$

$$y_p = \underbrace{\tfrac{1}{2}}_{\bar{C}_1}\sin t \underbrace{-\tfrac{1}{2}}_{\bar{C}_2}\cos t = R\cos(\underset{\substack{\uparrow \\ w=1}}{w}t - \delta) = \boxed{\tfrac{1}{\sqrt{2}}\cos\left(t+\tfrac{\pi}{4}\right)}$$

Steady state sol.

$$R = \sqrt{\bar{C}_1^2 + \bar{C}_2^2} = \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(-\tfrac{1}{2}\right)^2} = \sqrt{\tfrac{1}{2}} = \boxed{\tfrac{1}{\sqrt{2}}} \leftarrow \text{amplitude}$$

$$\tan\delta = \frac{\bar{C}_2}{\bar{C}_1} = -1 \quad \delta = -\frac{\pi}{4}$$

<u>Ex.2</u> A spring is stretched 4 in $\overset{L}{\ }$ by a mass that weighs 8 (lb). The mass is attached to a dashpot that has a damping const of $\frac{1}{4}\left(\frac{lb\cdot sec}{ft}\right)$ and is acted upon by an external force of 5 sin2t (lb). Find the DE that governs this motion.

$$mu'' + \gamma u' + ku = F(t).$$

$$m = \frac{\text{weight}}{g} = \frac{8\ lb\cdot sec^2}{32\cdot ft} = \frac{1}{4}(\text{slug}).$$

$$\gamma = \frac{1}{4}\left(\frac{lb\cdot sec}{ft}\right)$$

$$k = \frac{mg}{L} = \frac{8\ lb}{\frac{1}{3}(ft)} = 24\left(\frac{lb}{ft}\right), \quad F(t) = 5\sin 2t\ (lb).$$

$$L = 4(in) = \tfrac{1}{3}(ft)$$

$$\Longrightarrow \boxed{\tfrac{1}{4}u'' + \tfrac{1}{4}u' + 24u = 5\sin 2t}$$

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§6.1 Laplace Transform.

$$\int_a^\infty f(t)\,dt = \lim_{A\to\infty}\int_a^A f(t)\,dt \quad \text{improper integral}$$

$$|f(t)| \le g(t) \text{ and } \int_a^\infty g(t)\,dt < \infty \Rightarrow \int_a^\infty f(t)\,dt < \infty.$$

Def. $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$,

is called a <u>Laplace transform</u> of f.

Thm. 1) f - piecewise continuous fct on $0 \le t \le A$

2) $|f(t)| \le Ke^{at}$, $t \ge M$

then $\mathcal{L}\{f(t)\} = F(s)$ exists for all $s > a$.

Properties:

1) $\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1\} + c_2 \mathcal{L}\{f_2\}$

2) $\mathcal{L}\{f \cdot g\} \neq \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}$

3) $\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \lim\limits_{A\to\infty} \int_0^A e^{-st}\,dt =$

$= \lim\limits_{A\to\infty} \left(-\frac{1}{s} e^{-st}\right)\Big|_0^A = \lim\limits_{A\to\infty} \left(-\frac{e^{-sA}}{s} + \frac{1}{s}\right) = \frac{1}{s}$

$\Rightarrow \mathcal{L}\{3\} = 3 \cdot \mathcal{L}\{1\} = \frac{3}{s}$ as a corollary

4) $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at}\,dt = \lim\limits_{A\to\infty} \int_0^A e^{(a-s)t}\,dt$

$= \lim\limits_{A\to\infty} \left(\frac{1}{a-s} \cdot e^{(a-s)t}\right)\Big|_0^A = \lim\limits_{A\to\infty} \left(\frac{e^{A(a-s)}}{a-s} - \frac{1}{a-s}\right) = \frac{1}{s-a}$

assume $a-s<0$

$\boxed{s>a}$

5) $\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$

$\mathcal{L}\{f''\} = s^2 \mathcal{L}\{f\} - sf(0) - f'(0)$

....

$\mathcal{L}\{f^{(n)}\} = s^n \mathcal{L}\{f\} - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)$