Math 214.

Lecture 16.

Mechanical & Electrical Vibrations.

Spring-mass system: $\begin{cases} mu'' + \gamma u' + ku = F \\ u(0) = u_0 \\ u'(0) = u_0' \end{cases}$

Special cases.

Case I. Free undamped vibration:

Free $\to F = 0$, undamped $\to \gamma = 0$

$$\begin{cases} mu'' + ku = 0 \\ u(0) = u_0, \quad u'(0) = u_0' \end{cases} \qquad m \cdot r^2 + k = 0$$

$$r^2 = -\frac{k}{m} \Rightarrow r = \pm\sqrt{\frac{k}{m}} \cdot i$$

$$u(t) = C_1 \cos\sqrt{\tfrac{k}{m}}\,t + C_2 \sin\sqrt{\tfrac{k}{m}}\,t, \qquad \boxed{\omega_0 = \sqrt{\tfrac{k}{m}}} \text{ – natural frequency}$$

$$\boxed{T = \frac{2\pi}{\omega_0}} \text{ – natural period}$$

Equivalent form:

$$u(t) = R\cos(\omega_0 t - \delta) =$$
$$= R\cos(\omega_0 t)\cdot\cos\delta + R\sin(\omega_0 t)\cdot\sin\delta$$
$$= (R\cos\delta)\cdot\cos(\omega_0 t) + (R\sin\delta)\sin(\omega_0 t)$$

Originally: $u(t) = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t$

So $\left.\begin{array}{l} C_1 = R\cos\delta \\ C_2 = R\sin\delta \end{array}\right\}$ $\quad R^2\cos^2\delta + R^2\sin^2\delta = C_1^2 + C_2^2$

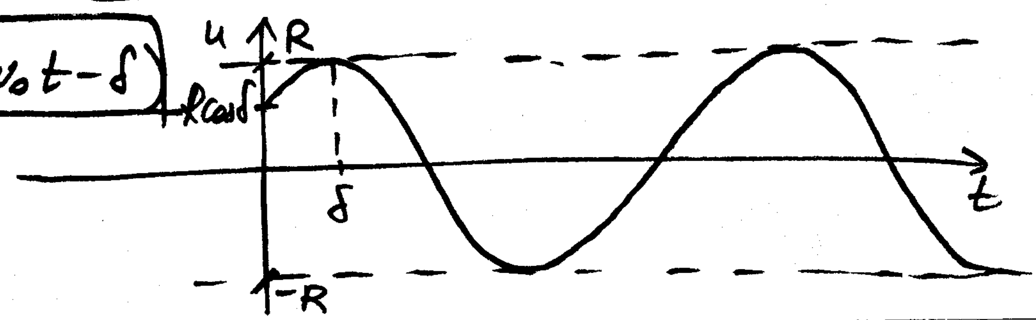
$$R^2 = C_1^2 + C_2^2$$

$$\boxed{R = \sqrt{C_1^2 + C_2^2}} \text{ – amplitude}$$

$$\frac{C_2}{C_1} = \tan\delta \Rightarrow \boxed{\delta = \arctan\left(\frac{C_2}{C_1}\right)} \text{ – phase shift}$$

$$\Rightarrow \boxed{u(t) = R\cos(\omega_0 t - \delta)}$$

Steady oscillation



Case III. (1) $\underset{F \neq 0}{\text{Forced}}$ $\underset{\gamma \neq 0}{\text{damped}}$ Vibration.

$$mu'' + \gamma u' + ku = F(t), \quad \boxed{F(t) = F_0 \cos \omega t}$$

assume forcing is periodic

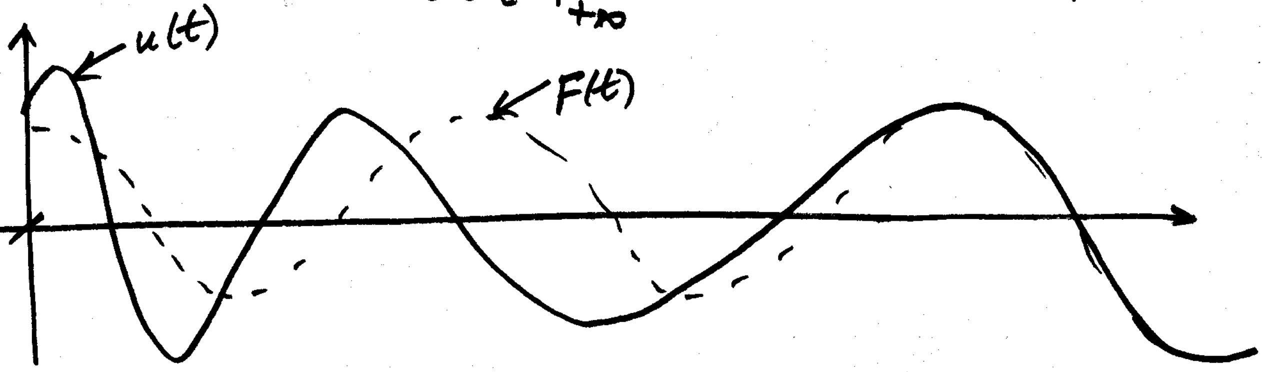
$$u(t) = \underbrace{\left[C_1 e^{-\frac{\gamma t}{2m}} \cos \mu t + C_2 e^{-\frac{\gamma t}{2m}} \sin \mu t\right]}_{\text{hom. eqn. solution}} + y_p$$

$$y_p = A \cos \omega t + B \sin \omega t = R \cos(\omega t - \delta)$$

$$\Rightarrow u(t) = \underbrace{C_1 e^{-\frac{\gamma t}{2m}} \cos \mu t + C_2 e^{-\frac{\gamma t}{2m}} \sin \mu t}_{\text{transient solution} \to 0 \text{ as } t \to +\infty} + \underbrace{R\cos(\omega t - \delta)}_{\text{steady-state solution}}$$

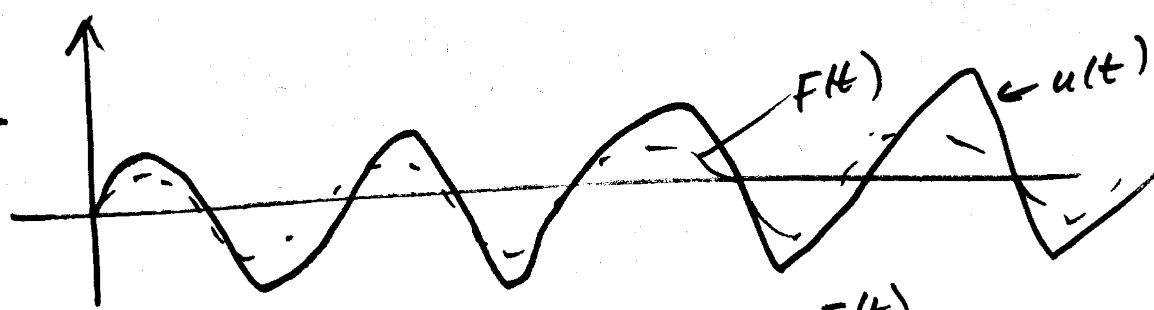
if $\boxed{\frac{\omega}{\mu} < 1}$

amplitude stays const

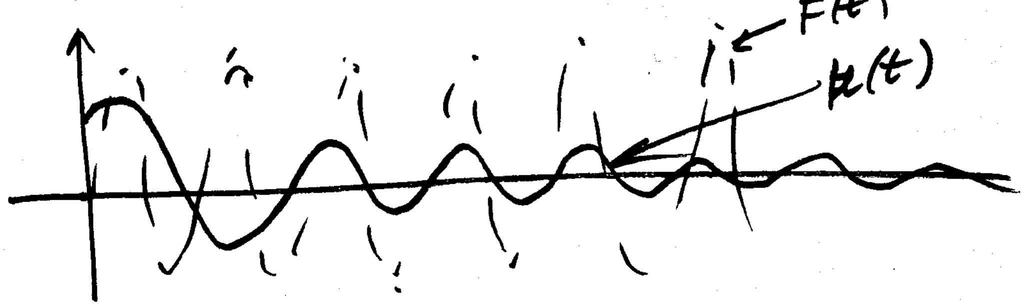


if $\boxed{\omega = \mu}$

amplitude will grow



if $\boxed{\frac{\omega}{\mu} > 1}$



Case III. (2) Forced free vibration.

$F \neq 0 \quad \gamma = 0.$

$$mu'' + ku = F_0 \cos \omega t$$

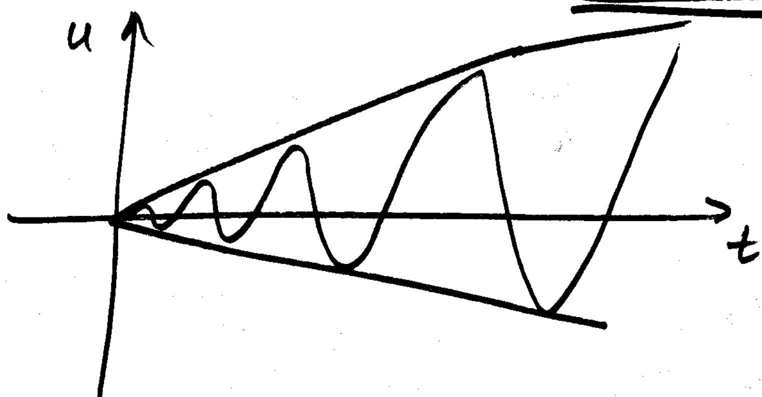
$$u(t) = \underbrace{(C_1 \cos \omega_0 t + C_2 \sin \omega_0 t)}_{\text{hom. solution.}} + y_p$$

Case A. $\boxed{\omega = \omega_0}$ $\boxed{\text{resonance}}$

We have duplication: $y_p = (A \cos \omega t + B \sin \omega t) \cdot t$

$$\Rightarrow u = R_1 \cos(\omega_0 t - \delta_1) + \underline{t \cdot R_2 \cos(\omega_0 t - \delta_2)}$$

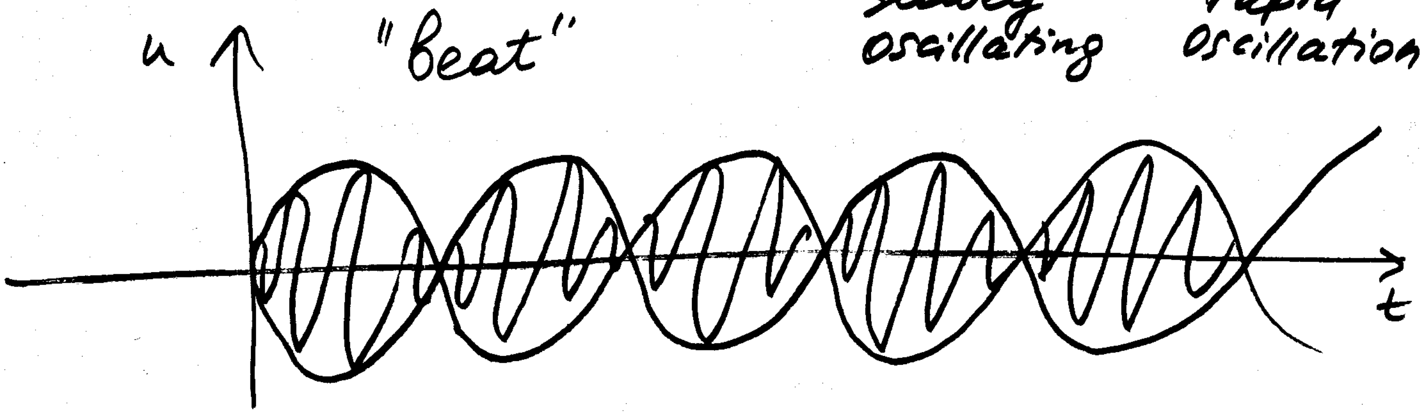
grows as $t \to +\infty$ infinitely large



Case B. $\boxed{\omega \neq \omega_0}$, $y_p = A \cos \omega t + B \sin \omega t$

$$u = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

If $\left.\begin{array}{l} u(0) = 0 \\ u'(0) = 0 \end{array}\right\} \Rightarrow u = \underbrace{\left[\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2}\right]}_{\text{slowly oscillating}} \underbrace{\sin \frac{(\omega_0 + \omega)t}{2}}_{\text{rapid oscillation}}$



Case II. Free damped vibration:

$F=0 \quad \gamma \neq 0$

$$mu'' + \gamma u' + ku = 0$$

$$mr^2 + \gamma r + k = 0$$

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$$

1) $\gamma^2 - 4km > 0 \Rightarrow$ 2 real distinct roots r_1, r_2

$$\boxed{u = C_1 e^{r_1 t} + C_2 e^{r_2 t}}$$

$r_{1,2} < 0$ since $-\gamma + \sqrt{\gamma^2 - 4km} < 0$

$\sqrt{\gamma^2 - 4km} < \gamma$

$\gamma^2 - 4km < \gamma^2$ - always true

So $\lim_{t\to+\infty} u(t) = 0$, but there are no oscillations $\Rightarrow$ $\boxed{\text{overdamped case}}$

2) $\gamma^2 - 4km = 0 \Rightarrow$ repeated real roots $r = r_1 = r_2 = -\frac{\gamma}{2m}$

$$\boxed{u = C_1 e^{-\frac{\gamma t}{2m}} + C_2 t e^{-\frac{\gamma t}{2m}}} \to 0 \text{ as } t \to +\infty \quad \text{since } r = -\frac{\gamma}{2m} < 0.$$

$\boxed{\text{Critical damping}}$

3) $\gamma^2 - 4km < 0 \Rightarrow$ 2 complex roots

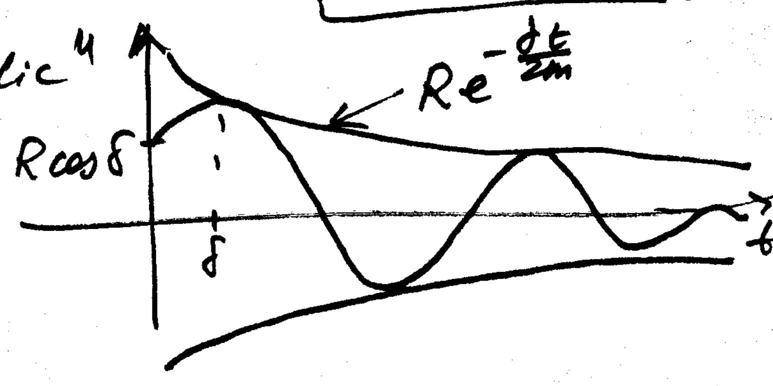
$$u = e^{-\frac{\gamma t}{2m}}(A\cos\mu t + B\sin\mu t) = \boxed{Re^{-\frac{\gamma t}{2m}}\cos(\mu t - \delta)}$$

$\boxed{\text{underdamped case}}$

$\mu = \frac{\sqrt{4km - \gamma^2}}{2m}$ - quasifrequency

$T = \frac{2\pi}{\mu}$ - quasiperiod

Oscillations are not periodic in strict sense since $u(t+T) \neq u(t)$



Ex. 1. A mass of 3 kg is attached to a spring with $k = 12\,\frac{N}{m}$.
What value of damping coeff will make the system critically damped?

$m = 3\,(kg)$

$k = 12\,\left(\frac{N}{m}\right)$

$\gamma_{cr}^2 - 4km = 0$

$\gamma_{cr} = \sqrt{4km} = 2\sqrt{km} = 2\sqrt{12 \cdot 3} = \boxed{12}$

$\frac{N}{m} \cdot kg$