Math 214.
Lecture 14

$$y'' + p(t)y' + q(t)y = g(t) \neq 0.$$

$$y_{\text{gen. nonhom.}}(t) = y_{\text{gen. hom}}(t) + \underbrace{y_p}_{\text{particular solution to nonhom. eqn.}}$$

Ex. $y'' - 2y' - 3y = \underline{\underline{3e^{2t}}}$ — nonhom.

$y'' - 2y' - 3y = 0$ — hom.

$r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0 \Rightarrow r_1 = 3, \; r_2 = -1$

$\underline{y_{\text{gen. hom.}} = C_1 e^{3t} + C_2 e^{-t}}$

Take $y_p = Ae^{2t}$ (by looking at the RHS).

Find A by plugging y_p into $y'' - 2y' - 3y = 3e^{2t}$

$$\left.\begin{aligned} y_p' &= 2Ae^{2t} \\ y_p'' &= 4Ae^{2t} \end{aligned}\right\} \Rightarrow \cancel{4Ae^{2t}} - \cancel{4Ae^{2t}} - 3Ae^{2t} = 3e^{2t}$$

$$-3Ae^{2t} = 3e^{2t}$$

$$A = -1$$

$$\Rightarrow \boxed{y_p = -e^{2t}}$$

$$\boxed{y_{\text{gen. nonhom.}}(t) = C_1 e^{3t} + C_2 e^{-t} - e^{2t}}$$

Ex. 2 $y'' - 2y' - 3y = \underline{\underline{3e^{3t}}}$

Let us try to take $y_p = Ae^{3t}$

$y_p' = 3Ae^{3t}$

$y_p'' = 9Ae^{3t}$

Recall $y_{gen.hom.} = C_1 e^{3t} + C_2 e^{-t}$

Plug y_P into nonhom. eqn. to get A:

$$9Ae^{3t} - 6Ae^{3t} - 3Ae^{3t} \neq 3e^{3t}$$
$$0 \neq 3e^{3t}$$

$\Rightarrow y_p = Ae^{3t}$ does not satisfy the nonhom. equation for any A

Since this in fact satisfies homogeneous equation.

<u>Remedy</u>: Take instead $\underline{y_p = Ate^{3t}}$

<u>Summary of method of undetermined coefficients</u>

for $\boxed{ay'' + by' + cy = g(t)}$

<u>Step 1.</u>

Solve $ay'' + by' + cy = 0$

(Find roots r_1, r_2 of char. eqn $ar^2 + br + c = 0$)

<u>Step 2.</u> Determine the type of $g(t)$. Find $y_p(t)$.

$g(t)$	$y_p(t)$
① "exp type": $e^{\alpha t}$ $\longrightarrow$	$Ae^{\alpha t} \cdot t^s$ s = multiplicity of α as a root of char. eqn. = # times it appears as a root of char. eqn.
② "trig type": $\left.\begin{matrix} \sin\beta t \\ \text{or } \cos\beta t \end{matrix}\right\}$ $\longrightarrow$	$(A\sin\beta t + B\cos\beta t) \cdot t^s$ s = # times $r = \pm\beta i$ appears as a root of char. equation

③ "poly type": $a_n t^n + a_{n-1} t^{n-1} + \ldots + a_0$

$(B_n t^n + \ldots + B_0) \cdot t^s$

$s = \#$ times 0 appears as a root of char. eqn.

To generalize (combining types ①-③):

$\underbrace{P_n(t)}_{\deg = n \text{ poly}} \cdot e^{\alpha t} \longrightarrow Q_n(t) e^{\alpha t} \cdot t^s$

$\left.\begin{array}{l} P_n(t) \cos\beta t \\ P_n(t) \sin\beta t \end{array}\right\} \longrightarrow Q_n(t)(A \sin\beta t + B\cos\beta t) t^s$

$\left.\begin{array}{l} P_n(t) e^{\alpha t} \sin\beta t \\ P_n(t) e^{\alpha t} \cos\beta t \end{array}\right\} \longrightarrow Q_n(t) e^{\alpha t}(A \sin\beta t + B\cos\beta t) \times t^s$

$s = \#$ times $r = \alpha \pm i\beta$ is a root of char. poly

If $g(t) = g_1(t) + \ldots + g_n(t) \longrightarrow$ add corresp. y_p's: $y_p = y_p^1 + \ldots + y_p^n$

ex: $e^{\alpha t} + \cos\beta t + P_n(t) \longrightarrow y_p = A e^{\alpha t} \cdot t^{s_1} + (B\sin\beta t + C\cos\beta t) \cdot t^{s_2} + Q_n(t) t^{s_3}$

t^s with $s > 0$ means there is a duplication of y_p with some part of the homogeneous solution.

Ex. $y''-y'-2y=3e^{-t}$

$y''-y'-2y=0$

$r^2-r-2=0$

$(r-2)(r+1)=0 \Rightarrow y_{gen.\,hom}=C_1e^{2t}+C_2e^{-t}$

$y_p=Ae^{-t}$ - duplicates hom. sol.

So need to modify by t: $y_p=Ate^{-t}$

Plug in to get A: $y_p'=Ae^{-t}-Ate^{-t}$

$$y_p''=-Ae^{-t}-Ae^{-t}+Ate^{-t}$$
$$=-2Ae^{-t}+Ate^{-t}$$

$$\Rightarrow (-2Ae^{-t}+Ate^{-t})-Ae^{-t}+Ate^{-t}-2Ate^{-t}=3e^{-t}$$

$$-3Ae^{-t}+(\cancel{2Ate^{-t}}-\cancel{2Ate^{-t}})=3e^{-t}$$

$$-3Ae^{-t}=3e^{-t} \Rightarrow A=-1 \Rightarrow \boxed{y_p=-te^{-t}}$$

$$\boxed{y_{gen.\,nonhom.}=C_1e^{2t}+C_2e^{-t}-te^{-t}}$$

Ex. $y''+4y'+4y=2e^{-2t}$

Find the form of y_p without solving for the unknowns.

$r^2+4r+4=0$

$(r+2)^2=0 \quad r_1=r_2=-2 \Rightarrow y_{hom}=C_1e^{-2t}+C_2te^{-2t}$

$$\boxed{y_p=Ae^{-2t}\cdot t^2}$$

Ex. $y''+4y'+4y=2te^{-2t}$

$y_p=(At+B)e^{-2t}\cdot t^2$

Ex. $y'' + 2y' - 3y = 5\sin 3t$

$r^2 + 2r - 3 = 0$

$(r+3)(r-1) = 0 \qquad y_{hom.} = C_1 e^{t} + C_2 e^{-3t}$

$y_p = A\sin 3t + B\cos 3t$ ← this works since no duplication

$y_p' = 3A\cos 3t - 3B\sin 3t$

$y_p'' = -9A\sin 3t - 9B\cos 3t$

$\Rightarrow \underline{-9A\sin 3t} - \underline{9B\cos 3t} + \underline{6A\cos 3t} - \underline{6B\sin 3t}$
$\underline{- 3A\sin 3t} - \underline{3B\cos 3t} = \underline{5\sin 3t}$

$(-12A - 6B)\sin 3t + (6A - 12B)\cos 3t = 5\sin 3t$

$(-12A - 6B - 5)\sin 3t + (6A - 12B)\cos 3t = 0.$

$\{\sin 3t, \cos 3t\}$ - indep. functions

$\Rightarrow$ there are no constants C_1, C_2 not both zero s.t. $C_1 \sin 3t + C_2 \cos 3t = 0.$

$$\Rightarrow \begin{cases} -12A - 6B - 5 = 0 \\ 6A - 12B = 0 \end{cases} \qquad 2\times \begin{cases} 12A + 6B = -5 \\ 6A - 12B = 0 \end{cases}$$

$30A = -10 \quad A = -\frac{1}{3}$

$B = \frac{1}{2}A = -\frac{1}{6}$

$\Rightarrow y_p = -\frac{1}{3}\sin 3t - \frac{1}{6}\cos 3t$

$$\boxed{y_{nonhom.}(t) = C_1 e^{t} + C_2 e^{-3t} - \frac{1}{3}\sin 3t - \frac{1}{6}\cos 3t.}$$