Math 214.

Lecture 13.

Summary (of const coeff 2nd linear homogeneous case)

$ay'' + by' + cy = 0 \Rightarrow ar^2 + br + c = 0$

Char. eqn. r_1, r_2 - roots

1) $r_1 \neq r_2$, real distinct

$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

2) $r = r_1 = r_2$ real repeated

$y = C_1 e^{rt} + C_2 t e^{rt}$

3) $r_{12} = \lambda \pm i\mu$ complex

$y = C_1 e^{\lambda t} \cos\mu t + C_2 e^{\lambda t} \sin\mu t$

Methods for obtaining 2nd linearly indep. solution having a 2nd order eqn and the first solution $y_1(t)$ available:

Reduction of order

Assume $y_2(t) = v(t) y_1(t)$
Plug into the eqn to get $v(t)$

Abel's Theorem.

$y'' + p(t)y' + q(t)y = 0$

$W(y_1, y_2)(t) = Ce^{-\int p(t)dt}$

Compute Wronskian

Then use the fact

$W(y_1,y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

to get y_2

Ex. 1 $t^2 y'' - 4ty' + 6y = 0$, $t > 0$, $y_1 = t^2$

Find ~~the~~ a second lin. indep. solution y_2.

Method 1: Abel's Theorem.

$$y'' - \frac{4}{t}y' + \frac{6}{t^2}y = 0 \qquad p(t) = -\frac{4}{t}$$

$$W(y_1, y_2) = Ce^{-\int\left(-\frac{4}{t}\right)dt} = Ce^{4\ln t} = Ct^4$$

by def $\parallel$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^2 & y_2 \\ 2t & y_2' \end{vmatrix} = t^2 y_2' - 2t y_2$$

$$\Rightarrow \quad t^2 y_2' - 2t y_2 = Ct^4$$

$$y_2' - \frac{2}{t}y_2 = Ct^2 \leftarrow \text{standard form}$$

$$p(t) = -\frac{2}{t}, \quad \mu(t) = e^{\int p(t)dt} = e^{-\int \frac{2}{t}dt} = e^{-2\ln t} = t^{-2}$$

$$(t^{-2}y_2)' = C$$

$$t^{-2}y_2 = Ct + C_1$$

$$y_2 = Ct^3 + C_1 t^2$$

standard choice: $C = 1,\ C_1 = 0 \Rightarrow \boxed{y_2 = t^3}$

Additional question: what is the gen. sol. of $t^2y'' - 4ty' + 6y = 0$?

Since $y_1 = t^2,\ y_2 = t^3$ are lin. indep. solutions $\Rightarrow$ they form a fund. set $\Rightarrow$ $\boxed{y = C_1 t^2 + C_2 t^3 \text{ is the gen. solution}}$

Method 2: Reduction of order.

$$y_1 = t^2 \Rightarrow y_2 = v(t)t^2$$

$$y_2' = v' \cdot t^2 + 2t \cdot v$$

$$y_2'' = v'' t^2 + 2tv' + 2tv' + 2v$$

Plug in: $t^2 y_2'' - 4t y_2' + 6y_2 = 0$

$$t^2(v''t^2 + 4tv' + 2v) - 4t(v' t^2 + 2tv) + 6v \cdot t^2 = 0$$

$$t^4 v'' + \underbrace{(4t^3 - 4t^3)}_{=0} v' + (\underbrace{2t^2 - 8t^2 + 6t^2}_{=0}) v = 0$$

$\Rightarrow t^4 v'' = 0 \qquad u = v' \leftarrow$ change of variables

$u' = v''$

$\Rightarrow t^4 u' = 0$

$t > 0 \Rightarrow u' = 0$

$u = C \Rightarrow v' = C$

$\underline{v = Ct + C_1}$

$y_2 = v(t) \cdot t^2 = (Ct + C_1)t^2 = Ct^3 + C_1 t^2$

Pick $C = 1,\ C_1 = 0$

$\Rightarrow \boxed{y_2 = t^3}$

Ex. 2. $t^2 y'' + 2ty' - 2y = 0$, $t > 0$, $y_1 = t$

$y_2 = v \cdot t = v(t) \cdot t$

$y_2' = v' t + v$

$y_2'' = v'' t + v' + v'$

$t^2 (v'' t + 2v') + 2t(v' t + v) - 2vt = 0$

$t^3 v'' + (2t^2 v' + 2t^2 v') + \underline{\cancel{2vt} - \cancel{2vt}} = 0$

$t^3 v'' + 4t^2 v' = 0$.

$u = v' \Rightarrow t^3 u' + 4t^2 u = 0$

$u' + \frac{4}{t} u = 0 \qquad \mu = e^{\int \frac{4}{t} dt} = t^4$

$(t^4 u)' = 0$

$t^4 u = C \Rightarrow u = Ct^{-4}$

$u = v' = Ct^{-4}$

$v = -\frac{C}{3} t^{-3} + C_1$

$y_2 = \left(-\frac{C}{3} t^{-3} + C_1\right) t = C_1 t - \frac{C}{3} t^{-2}$

Simplest form: $\boxed{y_2 = t^{-2}}$ $\qquad C_1 = 0$, $C = -3$

Ex. 3 $\boxed{t^2y''-t(t+2)y'+(t+2)y=0,\ t>0 \\ y_1=t \quad \text{Find } y_2(t).}$

Abel's Theorem: $y''\underbrace{-\frac{t+2}{t}}y'+\frac{t+2}{t^2}y=0$

$$W(y_1,y_2)=Ce^{-\int p(t)dt}=Ce^{\int\frac{t+2}{t}dt}=Ce^{t+2\ln t}$$
$$=Ct^2e^t$$

$$\| $$

$$\begin{vmatrix} t & y_2 \\ 1 & y_2' \end{vmatrix}=ty_2'-y_2$$

$$\Rightarrow\ ty_2'-y_2=Ct^2e^t$$

$$y_2'\underbrace{-\tfrac{1}{t}}_{p(t)=-\frac{1}{t}}y_2=Cte^t \qquad \mu(t)=e^{\int p(t)dt}=e^{-\ln t}=t^{-1}$$

$$\Rightarrow\ (t^{-1}y_2)'=Ce^t$$

$$t^{-1}y_2=Ce^t+C_1$$

$$y_2=Cte^t+C_1t$$

Pick $C_1=0$ $C=1$ $\Rightarrow$ $\boxed{y_2=te^t}$

§3.6. Nonhomogeneous equations.

$$y''+p(t)y'+q(t)y=g(t)\neq0$$

Fact 1: y_1, y_2 - solutions to $y''+p(t)y'+q(t)y=g(t)$ then y_1-y_2 solves $y''+p(t)y'+q(t)y=0$.

Fact 2: If $y(t)=C_1y_1(t)+C_2y_2(t)$ - gen. solution to homogeneous eqn $y''+p(t)y'+q(t)y=0$

Then $y_{\text{nonhom.}}(t)=C_1y_1(t)+C_2y_2(t)+\underbrace{y_p(t)}_{\text{Special sol. to nonhom. eqn.}}$

$$\boxed{\begin{array}{c}\text{Gen. sol.}\\ \text{nonhom. eqn.}\end{array}} = \boxed{\begin{array}{c}\text{Gen. sol.}\\ \text{hom. eqn.}\end{array}} + \boxed{\begin{array}{c}\text{Particular sol.}\\ \text{of nonhom. eqn.}\end{array}}$$

<u>Reason</u>: $Y_1 - Y_2$ solves hom. eqn.

$\Rightarrow Y_1 - Y_2 = c_1 y_1(t) + c_2 y_2(t)$

Choose $Y_2 = Y_p(t) \leftarrow$ particular sol. of nonhom. eqn.

$\Rightarrow Y_1(t) = c_1 y_1(t) + c_2 y_2(t) + Y_p(t)$

The whole task is to find Y_p.