Math 214.
Lecture 12

$ay'' + by' + cy = 0 \leftarrow$ 2nd order linear const coeffs

$ar^2 + br + c = 0 \leftarrow$ characteristic eqn

Roots r_1, r_2:

1) $r_1 \neq r_2$ - real $\Rightarrow$

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

2) $r_1 = r_2$
3) $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$ - complex

$\left.\begin{matrix} 2) \\ 3) \end{matrix}\right\}$ today

Complex roots. $\quad ar^2 + br + c = 0$

$$r_{1,2} = \lambda \pm i\mu$$

$$y_1 = e^{r_1 t} = e^{(\lambda + i\mu)t} = e^{\lambda t} e^{i\mu t} = e^{\lambda t}(\cos\mu t + i \sin\mu t)$$

$$y_2 = e^{r_2 t} = e^{(\lambda - i\mu)t} = e^{\lambda t} e^{-i\mu t} = e^{\lambda t}(\cos\mu t - i \sin\mu t)$$

[Euler formula: $e^{it} = \cos t + i \sin t$]

$$y_1 + y_2 = 2e^{\lambda t}\cos\mu t = 2u(t)$$

$$y_1 - y_2 = (2i)e^{\lambda t}\sin\mu t = (2i)v(t)$$

$\left.\begin{matrix} u(t) = e^{\lambda t}\cos\mu t \\ v(t) = e^{\lambda t}\sin\mu t \end{matrix}\right\}$ are also solutions to $ay'' + by' + cy = 0$

$$W(u,v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^{\lambda t}\cos\mu t & e^{\lambda t}\sin\mu t \\ \lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t & \lambda e^{\lambda t}\sin\mu t + \mu e^{\lambda t}\cos\mu t \end{vmatrix} =$$

$$= e^{\lambda t}\cos\mu t\left(\lambda e^{\lambda t}\sin\mu t + \mu e^{\lambda t}\cos\mu t\right)$$

$$- e^{\lambda t}\sin\mu t\left(\lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t\right)$$

$$= e^{2\lambda t}\mu\cdot\cos^2\mu t + e^{2\lambda t}\mu\sin^2\mu t = \mu e^{2\lambda t} \neq 0$$

if $\mu \neq 0$
true in complex case

So $\{u(t), v(t)\}$ are independent and form a fund. set of solutions $\Rightarrow$

$$\boxed{y(t) = C_1 e^{\lambda t}\cos\mu t + C_2 e^{\lambda t}\sin\mu t}$$ is the general sol.

Ex. $y'' - 2y' + 2y = 0$ find gen. sol.

$r^2 - 2r + 2 = 0.$

$$r = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm \sqrt{4}\cdot\sqrt{-1}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

$r = \underbrace{1}_{\lambda} \pm \underbrace{i}_{\mu}$ compare with $r = \lambda \pm i\mu$

$\Rightarrow \boxed{\begin{array}{l}\lambda = 1\\ \mu = 1\end{array}}$

Plug into gen. sol. formula:

$$y = C_1 e^t \cos t + C_2 e^t \sin t$$

Ex. $\begin{cases} y'' + 4y = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$ Find solution to this IVP.

$r^2 + 4 = 0$

$r^2 = -4 \quad r = \pm 2i \quad \lambda = 0 \quad \mu = 2$

$y = C_1 \cos 2t + C_2 \sin 2t \leftarrow$ gen. sol.

$y(0) = C_1 + C_2 \cdot 0 = 0$

$\Rightarrow C_1 = 0.$

$$\boxed{\begin{aligned} \sin 0 &= 0 \\ \cos 0 &= 1 \end{aligned}}$$

$$\boxed{y = C_2 \sin 2t = \tfrac{1}{2} \sin 2t}$$

$y' = +2C_2 \cos 2t$

$y'(0) = 2C_2 \cdot 1 = 1 \Rightarrow C_2 = \frac{1}{2}$

Ex. $\begin{cases} y'' + 2y' + 2y = 0 \\ y(0) = 2,\ y'(0) = 3 \end{cases}$

$r^2 + 2r + 2 = 0$

$r = \dfrac{-2 \pm \sqrt{4-8}}{2} = \dfrac{-2 \pm 2i}{2} = -1 \pm i$

$\lambda = -1 \quad \mu = 1$

$y = C_1 e^{-t} \cos t + C_2 e^{-t} \sin t$

$y(0) = C_1 = 2 \Rightarrow$

$y' = -C_1 e^{-t} \cos t - C_1 e^{-t} \sin t - C_2 e^{-t} \sin t + C_2 e^{-t} \cos t$

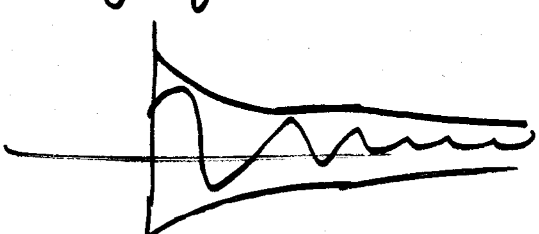
$y'(0) = -C_1 + C_2 = 3 \qquad \cancel{C_1 = 3C_2 - 3 = -1}$

$C_2 = 3 + C_1 = 5$

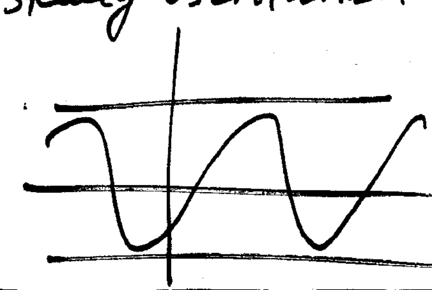
$[\, y = 2e^{-t} \cos t + 5e^{-t} \sin t \,]$

$$y = C_1 e^{\lambda t} \cos \mu t + C_2 e^{\lambda t} \sin \mu t$$

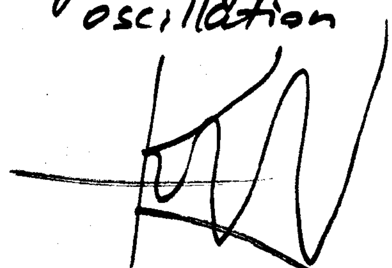
$\lambda < 0$
decaying oscillation



$\lambda = 0$
steady oscillation



$\lambda > 0$
growing oscillation



Last case: $r_1 = r_2 = r$ repeated root.

$ar^2 + br + c = 0$

$r = -\frac{b}{2a}$ is the location of the vertex where parabola touches the x-axis.

$y_1 = e^{-\frac{bt}{2a}}$ ← one solution of the DE $ay'' + by' + cy = 0$.

Need: second indep. solution.

Choose $y_2 = \underbrace{v(t)}_{\text{unknown } v(t)} e^{-\frac{bt}{2a}} = v(t)\, y_1(t)$ ← form of the second solution

Plug this into $ay'' + by' + cy = 0$.

$$y_2(t) = v \cdot e^{-bt/2a}$$

$$y_2' = v' e^{-bt/2a} - \frac{b}{2a} v e^{-bt/2a}$$

$$y_2'' = v'' e^{-bt/2a} - \frac{b}{2a} v' e^{-bt/2a} - \frac{b}{2a} v' e^{-\frac{bt}{2a}} + \left(\frac{b}{2a}\right)^2 v e^{-\frac{bt}{2a}}$$

$$a\left(v'' e^{-bt/2a} - \cancel{\frac{b}{a} v' e^{-bt/2a}} + \left(\frac{b}{2a}\right)^2 v e^{-bt/2a}\right)$$

$$+ b\left(\cancel{v' e^{-bt/2a}} - \frac{b}{2a} v e^{-bt/2a}\right) + cve^{-bt/2a} = 0$$

$$a v'' e^{-bt/2a} + e^{-bt/2a} \cdot v \cdot \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right) = 0 \quad \Big| \text{ cancel } e^{-bt/2a}$$

$$a v'' + v \cdot \left(c - \frac{b^2}{4a}\right) = 0 \implies \boxed{v'' = 0} \quad v' = c \quad \boxed{v = ct + c_1}$$

$$c - \frac{b^2}{4a} = \frac{4ac - b^2}{4a} = 0 \text{ since } \sqrt{b^2 - 4ac} = 0$$

in quadratic formula for repeated root case.

Hence $y_2 = te^{-bt/2a}$ is another lin. indep. sol.

$$\boxed{y = C_1 e^{-bt/2a} + C_2 t e^{-bt/2a}} \text{ gen. sol.}$$