Math 214.
Lecture 11.

Last time:

Thm 3 stated that if y_1, y_2 solve $y'' + p(t)y' + q(t)y = 0$, and $W(y_1, y_2)(t_0) \neq 0$, then there are constants C_1, C_2 such that $y(t) = C_1 y_1 + C_2 y_2$ solves the IVP

$$\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}$$

Thm 4. Moreover, if y_1, y_2 - solutions to $y'' + p(t)y' + q(t)y = 0$ (1) such that $W(y_1, y_2)(t_0) \neq 0$ for some point t_0.
Then $y(t) = C_1 y_1(t) + C_2 y_2(t)$ includes all solutions to equation (1).

Here $W(y_1, y_2)(t_0) \overset{\text{def}}{=} \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}$

$\{y_1, y_2\}$ is called a fundamental set of solutions.

Example! $y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$, $r_1 \neq r_2$ - real

Prove that $\{y_1, y_2\}$ is a fundamental set of solutions.

To show: $W(y_1, y_2)(t_0) \neq 0$ for some t_0.

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = r_2 e^{r_1 t} e^{r_2 t} - r_1 e^{r_1 t} e^{r_2 t} = (r_2 - r_1) e^{(r_1 + r_2)t} \neq 0$$

Example 2 Find $W(e^{2x}, 2e^{2x}) =$

$$= \begin{vmatrix} e^{2x} & 2e^{2x} \\ 2e^{2x} & 4e^{2x} \end{vmatrix} = 4e^{4x} - 4e^{4x} \equiv 0$$

$\Rightarrow \{e^{2x}, 2e^{2x}\}$ do not form a fundam. set

Ex. 3. $W(f,g) = 3e^{2x}$, $f = e^x \Rightarrow$ find $g(x)$.

$$W(f,g) \underset{def}{=} \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \boxed{\begin{vmatrix} e^x & g \\ e^x & g' \end{vmatrix} = 3e^{2x}}$$

$$e^x g' - e^x g = 3e^{2x}$$
$$g' - g = 3e^x$$

Integrating factor: e^{-x}

$$(e^{-x} g)' = 3$$
$$e^{-x} g(x) = 3x + C$$
$$\boxed{g(x) = 3xe^x + Ce^x}$$

Def. $f(t), g(t)$ are lin. dependent on I if there are C_1, C_2 not both zero such that $C_1 f(t) + C_2 g(t) = 0$ for all t in I.

$\Downarrow$

this is equivalent to $f(t) = -\frac{C_2}{C_1} g(t)$ if $C_1 \neq 0$

or to $g(t) = -\frac{C_1}{C_2} f(t)$ if $C_2 \neq 0$.

Linear dependence of f, g means f, g are proportional to each other.

Corollary:

either $\{y_1, y_2\}$ - lin. dependent on $I \Leftrightarrow W \equiv 0$ for all t in I

or $\{y_1, y_2\}$ - lin. indep. on $I \Leftrightarrow W \not\equiv 0$ for all t in I.

$\Leftrightarrow$ fund. set of Solutions: $\{y_1, y_2\}$

Ex.4 $t^2y'' - t(t+2)y' + (t+2)y = 0$ ⊛

Find $W(y_1, y_2)$ without solving the eqn.

Standard form: $y'' - \left(\frac{t+2}{t}\right)y' + \left(\frac{t+2}{t^2}\right)y = 0$

$$p(t) = -\frac{t+2}{t}, \quad q(t) = \frac{t+2}{t^2}$$

By Abel Thm:

$$W(y_1, y_2)(t) = Ce^{+\int \frac{t+2}{t}dt} = Ce^{\int(1+\frac{2}{t})dt}$$

$$= Ce^{t+2\ln t} =$$

$$= Ce^t \cdot e^{2\ln t} = Ce^t \cdot t^2$$

Suppose $y_1 = t$ is the first solution of ⊛.

Find ~~lin~~ second linearly indep. solution.

$$\begin{vmatrix} t & y_2 \\ 1 & y_2' \end{vmatrix} = Ce^t \cdot t^2$$

$$ty_2' - y_2 = Ce^t \cdot t^2$$

$$y_2' - \frac{1}{t}y_2 = Ce^t \cdot t$$

$$\mu = e^{-\int \frac{1}{t}dt} = e^{-\ln t} = t^{-1}$$

$$\left(\frac{1}{t}y_2\right)' = Ce^t$$

$$\frac{1}{t}y_2 = Ce^t + C_1$$

$$\boxed{y_2 = Cte^t + C_1 t}$$

$$y_2 = te^t \qquad \begin{matrix} C = 1 \\ C_1 = 0 \end{matrix}$$

Special choice that works.

Thm 5. If $W(f,g)(t_0) \neq 0$, t_0 belongs to I

then f, g - linearly independent on I.

Moreover, if $\{f, g\}$ - dependent on I $\Rightarrow$

$W(f,g)(t) = 0$ for all t in I.

Proof: Suppose f, g are dependent on I.

Then by def, $C_1 f(t) + C_2 g(t) = 0$ on I

for some C_1, C_2 not both zero.

Then $\begin{cases} C_1 f(t_0) + C_2 g(t_0) = 0 \\ C_1 f'(t_0) + C_2 g'(t_0) = 0 \end{cases}$ for some t_0 such that $W(f,g)(t_0) \neq 0$.

$$\begin{pmatrix} f(t_0) & g(t_0) \\ f'(t_0) & g'(t_0) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \text{ we get } \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ contradiction to $C_1, C_2 \neq 0$.

$A\vec{x} = \vec{0}$ system with $|A| = W(f,g)(t_0) \neq 0$.

So $\{f, g\}$ have to be independent.

Thm 6. (Abel's Thm).

y_1, y_2 - solutions to $y'' + p(t)y' + q(t)y = 0$

$p(t), q(t)$ continuous

then $\boxed{W(y_1, y_2)(t) = Ce^{-\int p(t)\,dt}}$

$(W' + p(t)W = 0)$

if $C = 0 \Rightarrow W \equiv 0$ for all t

if $C \neq 0 \Rightarrow W \neq 0$ for all t.