Math 214.
Lecture 10

§3.1-3.2 Second order equations

$$\begin{cases} ay'' + by' + cy = 0 \\ y(t_0) = 0 \\ y'(t_0) = 0 \end{cases}$$

in general
$$y'' = f(t, y, y')$$

→ RHS = 0
→ there is y'' (order = 2)
→ there is no explicit dep. on t
→ the form of the equation is linear
→ 2 initial conditions

⇒ **2nd order linear I.V.P with const coeffs**

We need 2 conditions since there are 2 integration consts to be resolved.

When solving a 2nd order ODE with const coeffs:

Step 1. Write down the char. equation

$$ar^2 + br + c = 0 \quad (*)$$

Step 2. Find the roots of $(*)$: r_1, r_2

Step 3. (a) If $r_1 \neq r_2$, real

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

general solution to the $ay'' + by' + cy = 0$.

(B) If $r = r_1 = r_2$, real (repeated root).

$$y = C_1 e^{rt} + C_2 t e^{rt}$$

(C) If $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, $i = \sqrt{-1}$

$$y = \cancel{C_1 u_1 e} \quad C_1 e^{\alpha t}\cos\beta t + C_2 e^{\alpha t}\sin\beta t$$

<u>Ex. 1</u> $\quad y'' + 3y' + 2y = 0$

Find gen. solution: Char. eqn $r^2 + 3r + 2 = 0$

$$(r+2)(r+1) = 0$$

$$r_1 = -2 \quad r_2 = -1$$

$$\Rightarrow y = C_1 e^{-2t} + C_2 e^{-t}$$

<u>Fundamental set of solutions</u>.

$$y'' + p(t)y' + q(t)y = g(t) \quad (1)$$

<u>Thm 3.</u> If $p(t), q(t), g(t)$ – cont. in I (interval), equation (1) has a unique sol. throughout interval I with any initial data $y(t_0) = y_0$, $y'(t_0) = y_0'$ that belongs to I.

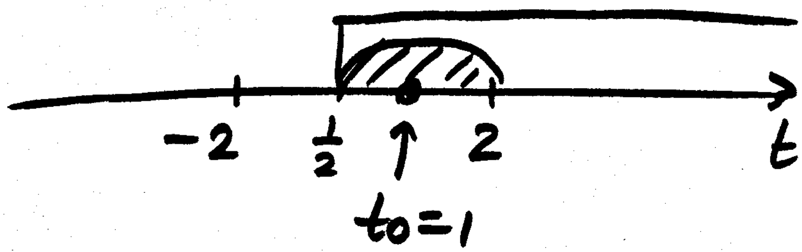
(Same as Thm 2 for 1st order eqns).

<u>Ex.</u> $\begin{cases} (t^2-4)y'' + (t+2)y' + \ln(2t-1)y = 0. \\ y(1) = 0,\ y'(1) = 2 \end{cases}$

What is the maximal interval where solution is unique?

Divide by (t^2-4):

$$y'' + \underbrace{\frac{t+2}{t^2-4}}_{p(t)} \cdot y' + \underbrace{\frac{\ln(2t-1)}{t^2-4}}_{q(t)} y = \underbrace{0}_{g(t)}.$$



$\Rightarrow \left(\frac{1}{2}, 2\right)$

Thm 2. y_1, y_2 - Solutions of (1): $y'' + p(t)y' + q(t)y = 0$

Then $c_1y_1 + c_2y_2 = y(t)$ - is also a solution to (1).

(This is only true for linear equs.).

$$\left.\begin{aligned} y(t) &= c_1y_1 + c_2y_2 \\ y' &= c_1y_1' + c_2y_2' \\ y'' &= c_1y_1'' + c_2y_2'' \end{aligned}\right\} \Rightarrow \begin{aligned} &(c_1y_1'' + c_2y_2'') + \\ &p(t)\cdot(c_1y_1' + c_2y_2') + \\ &q(t)(c_1y_1 + c_2y_2) = \end{aligned}$$

$$= c_1(\underbrace{y_1'' + p(t)y_1' + q(t)y_1}_{=0 \text{ since } y_1\text{-sol.}}) + c_2(\underbrace{y_2'' + p(t)y_2' + q(t)y_2}_{=0 \text{ since } y_2\text{-sol.}})$$

$= 0$ $\Rightarrow$ $y(t) = c_1y_1 + c_2y_2$ is also a solution of (1).

Thm 3.

$$\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = y_0, \\ y'(t_0) = y_0' \end{cases} \quad (*)$$

Suppose y_1, y_2 - solutions to (1) such that

$W(y_1, y_2)(t_0) = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \neq 0$

Then there are constants C_1, C_2 such that

$y = C_1 y_1(t) + C_2 y_2(t)$ is the solution to the IVP $(*)$.

Proof:

We know that if y_1, y_2 - solutions of (1),

$y = C_1 y_1 + C_2 y_2$ is also a sol. (By Thm 2).

We need to satisfy the initial conditions:

$$\begin{cases} y(t_0) = C_1 y_1(t_0) + C_2 y_2(t_0) = y_0 \\ y'(t_0) = C_1 y_1'(t_0) + C_2 y_2'(t_0) = y_0' \end{cases}$$

Kramer's rule:

$$C_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}, \quad C_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

If $W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} \neq 0$

$\Rightarrow$ we have a unique solution to the IVP $(*)$.

Def. $W(y_1, y_2) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$ is called Wronskian.

<u>You need to be able to:</u>

1) Find Wronskians for any y_1, y_2
2) Find solutions to IVPs by forming $y = C_1 y_1 + C_2 y_2$, where $W(y_1, y_2) \neq 0$