Math 685.
Lecture 8.

Eigenvalue problems: $Ax = \lambda x$

Methods:

1) Power method: $x_{k+1} = A x_k$

→ has to be scaled $\frac{x_k}{\|x_k\|}$ at every step to avoid possible overflow/underflow

→ linear convergence, dep. on $C = \left|\frac{\lambda_2}{\lambda_1}\right|$

$\lambda_1 > \lambda_2 \geq \dots \geq \lambda_n$

$e_{n+1} \sim C \cdot e_n$

→ x_0 has to be chosen s.t.

$x_0 = \underbrace{\alpha_1}_{\neq 0} v_1 + \dots + \alpha_n v_n$

→ can get complex eigenvalues, but has to have x_0 - complex in this case.

2) $x_{k+1} = (A - \sigma I) x_k$ ← shifted power iter.

$x_k \to$ max eigenvalue of $A - \sigma I$

$= \lambda_{max}(A) - \sigma$

3) Inverse shifted:

$x_k = (A - \sigma I) x_{k+1}$ ⇒

converges to $\lambda_{min}(A - \sigma I) = \min(\lambda_i - \sigma)$

where $\lambda_1, \dots \lambda_n$ - eig. values of A

↳ Have to solve a linear system of the form $(A - \sigma I)\underline{x} = B$.

4) <u>Rayleigh Quotient iteration</u>

$$\lambda = \frac{x^T A x}{x^T x}$$

<u>Step 1</u>: $\sigma_k = \dfrac{x_{k-1}^T A x_{k-1}}{x_{k-1}^T x_{k-1}}$

approximation to λ.

<u>Step 2</u>: $(A - \sigma_k I) y_{k+1} = x_k$,

$$\frac{y_{k+1}}{\|y_{k+1}\|} = x_{k+1}$$

$\oplus$ at least <u>quadratic</u> convergence
$e_{n+1} \sim C \cdot e_n^2$

$\ominus$ recomputing σ_k, factoring $A - \sigma_k I$ at every step is time consuming, but may be worth it due to savings from convergence speed.

5) Deflation: $HAH^{-1} = \begin{bmatrix} \lambda_1 & B^T \\ 0 & B \end{bmatrix}$

reduces problem to dim $= n-1$ and repeat

Need to have: good estimate of λ_1

$\ominus$ forming H is costly.

6) Simultaneous iteration

$$X_{k+1} = A X_k$$

have to normalize (same as in power iter.)

9) QR with preliminary reduction.

Step 1. Converts A to Hessenberg form

Step 2 Do QR iteration on Hessenberg

$\Rightarrow O(n^2)$ operations

if A - symmetric $\Rightarrow$ tridiagonal

$O(n)$ operations.

⊕ QR is very robust, esp. if all eigenvalues need to be computed.

⊖ Very expensive for large n

⊖ if only several $\lambda_i(A)$ are needed, might be outperformed by other methods.

⊖ ~~so~~ "fill-in" can occur for sparse A.

Krylov subspace methods.

Instead of working on the orthonormal basis of $\mathbb{R}^n$, construct approximations incrementally.

$$K_n = [\vec{x}_0 \; A\vec{x}_0 \; \dots \; A^{k-1}\vec{x}_0]$$

Want: rotations transforming A into a Hessenberg form though

$$K_n^{-1} A K_n = C_n \; (\text{Companion matrix})$$

<u>Alternative</u>:

Orthogonal iteration:

reduced QR-decomp.
$$\begin{cases} \hat{Q}_k R_k = X_{k-1} \\ X_k = A\hat{Q}_k \end{cases} \quad Q_k - n\times p \quad R_k = \begin{array}{c} \; p \\ p \begin{pmatrix} \diagdown\!\!/\!/\!/ \end{pmatrix} \end{array}$$

By Schur decomposition, there is a $\hat{Q}$ s.t. $A\hat{Q} = \hat{Q}B$ where B is in triangular form.

7) QR iteration: $\underline{A_k = R_k \hat{Q}_k}$, $A_{k-1} = \hat{Q}_k R_k$

$$A_k = \hat{Q}_k^H A_{k-1} \hat{Q}_k \longrightarrow \begin{pmatrix} \diagdown\!\!/\!/\!/ \\ 0 \end{pmatrix} \text{ for general } A$$

$\nwarrow$ unitary similar to A

$$\longrightarrow \begin{pmatrix} \diagdown & 0 \\ 0 & \diagdown \end{pmatrix} \text{ for } A\text{-symm.}$$

In general, can be slow to converge.

8) QR with shift:

$Q_k R_k = A_{k-1} - \sigma_k I \leftarrow$ QR on shifted A_{k-1}

$A_k = R_k Q_k + \sigma_k I \leftarrow$ recompute A_k

$\oplus$ speed up QR by using σ_k approximation

$$\sigma_k \approx a_{nn}^{(k-1)} \quad A_{k-1} = \begin{pmatrix} * & \\ & a_{nn} \end{pmatrix}$$

other shifts are possible

$\ominus$ QR requires $O(n^3)$ operations.

$K_n = Q_n R_n$

$H \equiv Q_n^H A Q_n$ – Hessenberg form

↑ want to compute elements of H

Arnoldi iter.

$$\begin{cases} 1) \; A q_k = h_{1k} q_1 + \dots + h_{k+1,k} \underline{\underline{q_{k+1}}} \\ 2) \; h_{jk} = q_j^H A q_k \end{cases}$$

By repeating this process: $A \rightsquigarrow H$

s.t. $\lambda(H)$ = Ritz values $\approx \lambda(A)$

$Q_k v(H)$ = Ritz vectors $\approx v(A)$

↑ eig. vectors of H ↑ eig. vec. of A

In case A–symm. or Hermitian ⇒ Arnoldi iteration can be made more efficient by using Lanczos iteration which converts A into a tridiagonal form: