Methods for least squares approximation

1) Normal eqns

\[ A^T A x = A^T y \]  \text{fastest}

\[ y = c_0 + c_1 x + \ldots + c_n x^n \]

\[ y_1 = c_0 + c_1 x_1 + \ldots + c_n x_1^n \]

\[ \vdots \]

\[ y_m = c_0 + c_1 x_m + \ldots + c_n x_m^n \]

\[ \begin{bmatrix}
1 & x_1 & \ldots & x_1^n \\
\vdots & \vdots & \ddots & \vdots \\
1 & x_m & \ldots & x_m^n 
\end{bmatrix}
\begin{bmatrix}
c_0 \\
c_1 \\
\vdots \\
c_n 
\end{bmatrix}
= 
\begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_m 
\end{bmatrix}

\text{if } (A^T A)^{-1} \text{ exists (full rank problem, no rank deficiency)}

\text{then normal eqns provide unique solution}

\Rightarrow \text{non-accurate answers}

2) Augmented system

\[ \begin{pmatrix}
I & A \\
A^T & 0
\end{pmatrix}
\begin{pmatrix}
x \\
r
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

\[ \Rightarrow r + A x = 0 \]

more robust, not as sensitive

storage requirements are high
3) **Transformations:**

   have to preserve the norm

   \[ \| P(Ax - b) \|_2 = \| Ax - b \|_2 = \| r \|_2 \]

   \( \Rightarrow P \) has to be orthogonal: \( P^T P = I \).

**(A) QR-decomposition:**

\[ A = Q \cdot R \]

\[ R = \begin{bmatrix} c \end{bmatrix} \]

\[ Q = \begin{bmatrix} \overline{Q_1} \\ \overline{Q_2} \end{bmatrix} \]

\[ R = \begin{bmatrix} \overline{R_1} \\ \overline{R_2} \end{bmatrix} \]

\[ R_1 \times x = c \]

\[ \text{reduced form} \]

\[ \text{Since } \| Qy \| = \| y \| \]

\[ A = Q \begin{bmatrix} R \\ 0 \end{bmatrix} = (Q_1 \ Q_2) \begin{bmatrix} R_1 \\ 0 \end{bmatrix} = Q_1 R \]

\[ \text{rank}(A) = n \Rightarrow \text{full rank, unique sol} \]

\[ \text{rank}(A) < n \Rightarrow \text{rank-deficiency} \]

\[ R \text{ becomes singular} \]

\( \Rightarrow \) column-pivoting is required.
Methods for finding $Q$:

1. Householder
2. Givens
3. Gram-Schmidt

\[ H = I - 2 \frac{v \cdot v^T}{v^T v} \]
\[ \| Hv \| = \| u \| \]
\[ Hv = u \cdot e_i \]

$H$ reflects in $\text{span}(A)^T$ so that all or some of vector components become zero, except for pivoting pivoting.

\[ v = \text{sign}(x_i) \cdot \|x\| e_1 + x \]

then

\[ \hat{x} = \pm \|x\| e_1 \]

\[ a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \xrightarrow{H} \begin{bmatrix} d \\ 0 \\ 0 \end{bmatrix} \]

\[ (I - 2 \frac{v \cdot v^T}{v^T v}) \cdot a = d \cdot e_1, \quad \leftrightarrow \quad e_1 = \text{first column of} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ a - 2 \frac{v \cdot a}{v^T v} = d \cdot e_1 \]

\[ \times \left( \frac{v^T u}{v^T a} \right) \]
\[
\frac{v^* v}{2 v^* a} \cdot (a - d e_i) = v^*
\]

Choose \( \nabla = a - d e_i \)

Since \( \|v\| = \|d e_i\| \Rightarrow \|v\| = \|a\| \)

\[
H = I - 2 \frac{v v^T}{v^T v}
\]

\( \|a\| = \sqrt{3} \)

\[\alpha = -\text{sign}(a) \cdot \|a\| \]

with \( v = (1) - (0) = (1, 0) \)

\( \alpha = -\sqrt{3} \)

\[
(1 + \sqrt{3})
\]

\[d) \quad \text{Given rotation:} \]

\[
\begin{pmatrix}
c, s \\ -s, c
\end{pmatrix}
\]

\[c = \frac{a_1}{\sqrt{a_1^2 + a_2^2}} \quad s = \frac{a_2}{\sqrt{a_1^2 + a_2^2}}
\]

\[\tan \theta = \frac{s}{c} = \frac{a_2}{a_1}
\]

\( \theta \) angle of rotation

\[v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{G} \begin{pmatrix} \pi \\ 0 \\ 0 \end{pmatrix}
\]

(1) rotate \( 1 \& 2 \)

\[G_1 v = \begin{pmatrix} a_1 \\ 1 \\ 0 \end{pmatrix}
\]

\[\|G_1 v\| = \|v\|
\]

(2) rotate \( 1 \& 3 \)

\[G_2 G_1 v = \begin{pmatrix} a_2 \\ 0 \\ 0 \end{pmatrix}
\]

\[G_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}
\]

\[G_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}
\]