# Math 685. Lecture 3

## Problem 3 HW2.

$\sin(x)$ – exact

$\sin(x+h)$ – perturbed value

$$\text{Cond } \# = \frac{\text{rel. change in } f(x)}{\text{rel. change in } x} = \frac{|\sin(x+h) - \sin(x)| \cdot |x|}{|\sin(x)| \cdot |h|}$$

$$\approx \frac{|\sin'(x)| \cdot \cancel{|h|} \cdot |x|}{|\sin(x)| \cdot \cancel{|h|}} = \frac{\cancel{\sin'}|\cos x| \cdot |x|}{|\sin x|}$$

if $x \to 0$ $\sin x \sim 0$ $\frac{\sin x}{x} \to 1$ (ok)

if $x \sim \pi k$ $\sin x \to 0$ $\frac{\sin x}{x} \not\to 1$ cond $\to \infty$

$k \neq 0.$

very sensitive to perturbations at $x = \pi k$, $k \neq 0$.

## Problem 4 HW2.

$ax^2 + bx + c = 0$ $\quad a = 1.22$, $b = 3.34$, $c = 2.28$

$\beta = 10$ $\quad p = 3$

$b^2 - 4ac$ : $\quad b^2 = 1.11556 \cdot 10^1 \sim 1.12 \cdot 10^1$

$+4ac = 1.11264 \cdot 10^1 \sim 1.11 \cdot 10^1$

Exact computation: $\quad 0.0292 = \underline{\underline{2.92 \cdot 10^{-2}}}$

In $p=3$ arithmetic: $\quad b^2 - 4ac \sim 0.01 \cdot 10^1 = \underline{1.0 \cdot 10^{-1}}$

Rel. error : $6.575 \cdot 10^{-1} \approx 60\%$

Problem 5 HW2

Stable form of an expression :
a form where numerical errors due to loss of significance, underflow and overflow are avoided as much as possible.

$x - \sin(x)$ at $x$ close to 0 gives troubles for rel. large $x$ ( $x >> N_{underflow}$ )
$\Rightarrow$ pay attention to subtractive cancellation first.

(a) $$\frac{\sin(x)\cdot(x+\sqrt{x^2-1})}{(x-\sqrt{x^2-1})(x+\sqrt{x^2-1})} = \sin(x)(x+\sqrt{x^2-1})$$

$$x^2-(x^2-1)=1$$

(c) $$\frac{(1-\sin(x))(1+\sin x)}{1+\sin x} = \frac{1-\sin^2 x}{1+\sin x} = \frac{\cos^2 x}{1+\sin x}$$

$$1-\sin(x) = 2\cos^2\frac{x}{2}$$

$1-\sin x$

Taylor at $x_0 = \frac{\pi}{2}$: $\sin x \approx \sin\frac{\pi}{2} + \cos(\frac{\pi}{2}) \cdot x + \ldots$

$$f(x) \approx f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\cdot\frac{(x-x_0)^2}{2!} + \ldots$$

$$\sin x \approx 1 + \underbrace{\cos\frac{\pi}{2}}_{0}\cdot(x-\frac{\pi}{2}) - \underbrace{\sin\frac{\pi}{2}}_{1}\cdot\frac{(x-\frac{\pi}{2})^2}{2!} + \ldots$$

$$1-\sin x \approx \frac{(x-\frac{\pi}{2})^2}{2} + \ldots$$

FLOP – floating point arithmetic

$$\boxed{\begin{array}{l} d_0 . d_1 d_2 \ldots d_{p-1} \cdot \beta^E \\ \neq \\ 0 \end{array}}$$

To add/subtract/multiply/divide:

1) align exponents
2) carry out add/subtract/multiply operation on mantissas
3) normalize

Significant digits: $\underbrace{1.002140}_{6 \text{ sign. digits}}0 \ldots 0$

## Pitfalls

1) Loss of significance due to subtraction of 2 similar numbers
2) Add large & small number

$x_1 = 50000 \rightarrow 5.0 \cdot 10^4 \rightarrow 5.000 \cdot 10^4$

$x_2 = 0.1 \rightarrow 1.0 \cdot 10^{-1} \rightarrow 0.00001 \cdot 10^4$

align exponents

$p = 5$

$$\begin{array}{r} 5.0000|0 \\ +\ 0.0000|1 \\ \hline 5.0000 \end{array}$$

small # gets lost

3) $A = 1 \cdot 10^{-6}$
$B = 2 \cdot 10^{-6}$
$C = 3 \cdot 10^{7}$

$(BC)A \neq (AB)C$

exponent with 1 digit

$(BC)A = 6 \cdot 10^{-5}$

$(AB)C = (\underset{\overset{\|}{0}}{2 \cdot 10^{-12}}) \cdot C = 0$

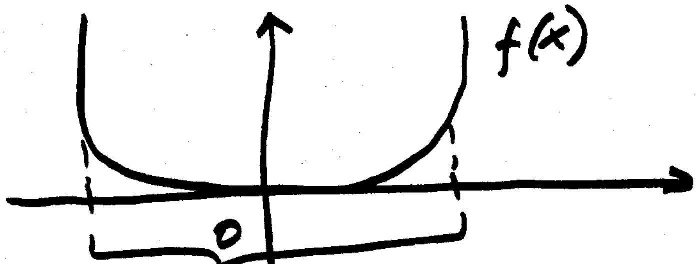
Multiply large numbers first.

Accuracy = stability + conditioning

Error estimate: $Ax = b$, $\tilde{x} = x + \Delta x$ :

$$\frac{\|\Delta x\|}{\|x\|} \le \mathrm{cond}(A) \cdot \frac{\|r\|}{\|A\| \cdot \|x\|}$$

$r = b - A\tilde{x}$ - residual




small residual, but large error