Math 685.
Lecture 2.

Linear systems.

$Ax = b$

Ex. $\begin{cases} x_1 + 2x_2 = 1 \\ 2x_1 + x_2 = 2 \end{cases}$ $\quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $\quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\det A = \det \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = 1 - 4 = -3 \neq 0$

nonsingular

$2\times \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 1 & 2 \end{array}\right) \underset{\text{ERO}}{\sim} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -3 & 0 \end{array}\right) \uparrow$

Gaussian elimination

backsubstitution

$-3 \cdot x_2 = 0 \Rightarrow x_2 = 0$

$x_1 + 2 \cdot 0 = 1 \Rightarrow x_1 = 1$

Solution: $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\operatorname{Ker} A = (\vec{x} \mid A\vec{x} = \vec{0})$

$\operatorname{Span} A = (A\vec{x}) \quad \forall \vec{x} \in \mathbb{R}^n$

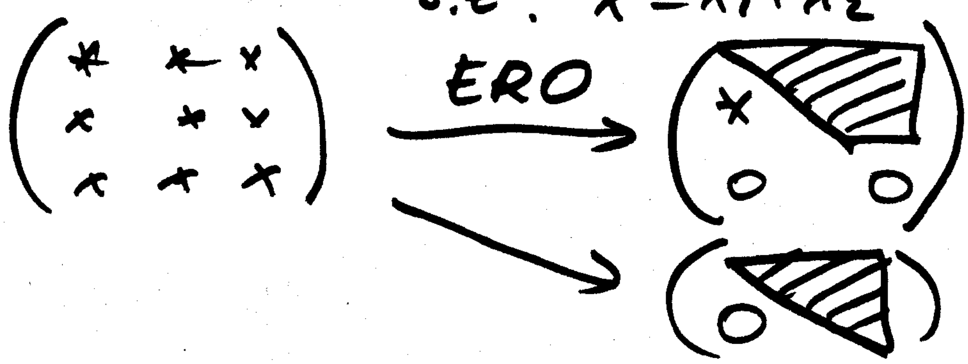
$\mathbb{R}^n = \operatorname{Ker}(A) \oplus \operatorname{Span}(A)$

direct sum

$\forall x \in \mathbb{R}^n \ \exists!\ x_1 \in \operatorname{Ker}(A)$

$x_2 \in \operatorname{Span}(A)$

s.t. $x = x_1 + x_2$



rank(A)<n

rank(A)=n

Ex. 3

$$\begin{bmatrix} \delta & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad 0 < \delta \ll 1$$

$$\delta < \frac{\varepsilon_{mach}}{2}$$

Machine answer by Gaussian elimination w/o pivoting:

$$\left[\begin{array}{cc|c} \delta & 1 & 1 \\ 0 & 1-\frac{1}{\delta} & -\frac{1}{\delta} \end{array}\right] \Longrightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 1 \end{array}$$

$$x_2 = \frac{-\frac{1}{\delta}}{1-\frac{1}{\delta}} = 1$$

Exact answer:

$$\left\{\begin{array}{l} x_1 = -\dfrac{1}{1-\delta} \\ x_2 = \dfrac{1}{1-\delta} \end{array}\right\}$$

Chose an <u>unstable algorithm</u> to work on ~~an~~ a well-conditioned problem. (Stability problem).

<u>With pivoting:</u>

$$\delta \times \begin{bmatrix} 1 & 1 \\ \delta & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1-\delta & 1 \end{array}\right]$$

good accurate answer $\Longleftarrow$

$$x_2 = \frac{1}{1-\delta}$$

$$x_1 = -\frac{1}{1-\delta}$$

Ex. 4

$$\begin{pmatrix} 0.661 & 0.991 \\ 0.5 & 0.75 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.330 \\ 0.25 \end{pmatrix}$$

True solution: $x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Ex. 2 $A = \begin{array}{cc} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} & \begin{array}{l} \to 3 \\ \to 7 \end{array} \\ \begin{array}{cc} \downarrow 4 & \downarrow 6 \end{array} & \end{array}$

$\|A\|_1 = \max\{\text{col. sums}\} = 6$

$\|A\|_\infty = \max\{\text{row sums}\} = 7$

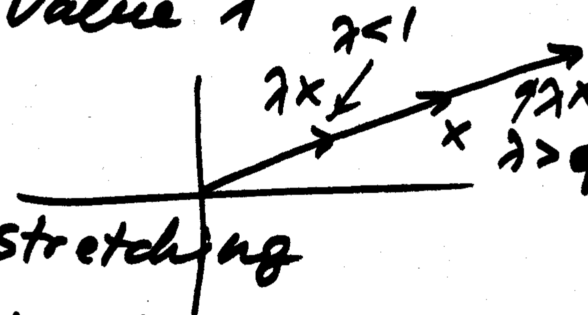
$\|A\| = \sup\limits_{x \neq 0} \dfrac{\|Ax\|}{\|x\|}$ - stretching of a vector by A

Consider x - eig. vector of A with eig. value λ

$Ax = \lambda x$

$\|Ax\| = \lambda \cdot \|x\|$ ← stretching

$A^{-1}x = \frac{1}{\lambda} x$ ← shrinking



$\operatorname{cond} A = \|A\| \cdot \|A^{-1}\|$ = ratio of stretching vs shrinking

$A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\|A\| = \|A^{-1}\| = 1$

$\operatorname{cond} A = 1$

Otherwise $\operatorname{cond}(A) > 1$.

Gaussian elimination with pivoting, in 3-sign. digit precision, we get $\tilde{x} = \begin{bmatrix} -0.47 \\ 0.647 \end{bmatrix}$

$$r = \cancel{Ax}\ A\tilde{x} - b = \begin{bmatrix} -0.00507 \\ -0.00025 \end{bmatrix}$$

Residual is very small, but error is still big due to ill-conditioning of the problem.