MATH 685
Lecture 6

Applied Math seminar:
Fri 1:30 - 2:30pm Research I, 301

Project 2

1. $y = C_1 e^{C_2 t}$
   $\log y = \underbrace{\log C_1}_{K} + \underline{C_2 t}$

   $y = te^{C_2 t} + C_3$
   $\log(a+b) \neq \log a + \log b$
   $\log te^{C_3 t} + \log C_3$

2. SVD for least squares
   $A = USV^T$, $A^+ = VS^+U^T$
   $Ax \cong b$ $\quad x = A^+b$ is the least squares solution.

3. QR
   $$A = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix} \leftarrow \text{in textbook}$$
   "qR(A)" command gives $R_{\text{matlab}} = \begin{bmatrix} R \\ 0 \end{bmatrix}$

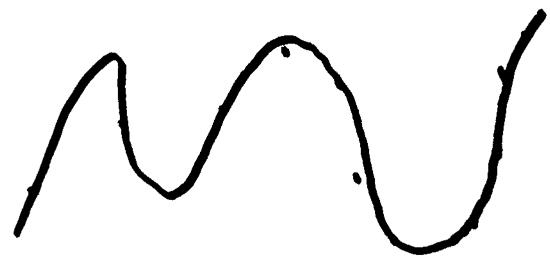
§7. Interpolation.

$(t_i, y_i) \leftarrow$ data pts

Want: $p(t)$ s.t. $p(t_i) = y_i$

1) discrete (t_i, y_i)
2) continuous $(t_i, f(t_i))$

Interpolant : → easy to use
→ represent data



$\varphi_i(t) \leftarrow$ basis functions.

<u>1 step</u> : Choose type of interpolation
→ poly or piecewise poly
→ trig
→ exp
→ rational

⇒ defines the space you will work in.

e.g. $\mathbb{P}_5$ is a space of polys of $\boxed{\deg \le 4}$

$$p_1 = t^4 + t^3 - t^2 \Rightarrow p_1 - p_2 = -2t^2$$
$$p_2 = t^4 + t^3 + t^2$$

<u>2 step</u> : Basis

Same space but different basis results in different numerical problem.

$\mathbb{R}^2$ $(0,1), (1,0)$

e.g. monomials:
basis in $\mathbb{P}_k$: $1, t, t^2, t^3 \ldots t^{k-1}$
is not optimal.

(t_i, y_i) $\quad p(t_i) = y_i$

$p(t) = \sum_{j=1}^{n} x_j \varphi_j(t)$ — representing interpolant as a linear comb. of basis function.

$p(t_i) = \sum_{j=1}^{n} x_j \varphi_j(t_i) = y_i$, $i = 1, \dots m$.

$A = (a_{ij}) = (\varphi_j(t_i)) \Rightarrow A\vec{x} = \vec{y}$

Coefficient $\quad$ given vector

Everything depends on the properties of A.

Choices of classes/Bases:

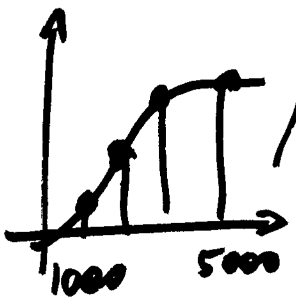
① Monomials : $\quad A$ = Vandermonde matrix

$$A = \begin{bmatrix} 1 & t_1 & t_1^{n-1} \\ \vdots & \vdots & \vdots \\ 1 & t_n & t_n^{n-1} \end{bmatrix}$$

$Ax = y$

$Az = 0 \Rightarrow z = 0$

$P_z(t) = z_1 + z_2 t + \dots + z_n t^{n-1}$



n pts $\quad t_1, \dots t_n \rightarrow P_z(t_i) = 0$

$\Rightarrow$ n roots

deg $\leq n-1$

Fundam. Thm of Calculus

$\Rightarrow P_z(t) \equiv 0 \Leftrightarrow \vec{z} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

A - nonsingular

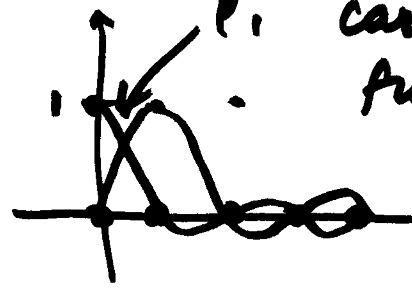
$\Rightarrow$ unique sol. to $Ax = y$.

cond(A) grows with n

$\Rightarrow$ although $Ax - b = r$ can be small actual values of $\vec{x}$ will carry some error.

2) Lagrange basis functions.

Lagrange cardinal functions



$$\varphi_j(t_i) = l_j(t_i) = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

$\uparrow$ basis function

$l_1(x_1) = 1$

$l_1(x_2) = l_2(x_3) = \ldots 0$

$$A = (\varphi_j(t_i))_{i,j} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$l_j(t_i) = \frac{(t-t_1)\cdots(t-t_{j-1})(t-t_{j+1})\cdots(t-t_n)}{(t_j-t_1)\cdots(t_j-t_{j-1})(t_j-t_{j+1})\cdots(t_j-t_n)}$$

$t_i = t_j \Rightarrow l_j(t_j) = 1.$

$p(t) = y_1 l_1(t) + \ldots + y_n l_n(t)$

interpolating diffent $\vec{y}$ is easy.

But: calculating l_j is hard.

3) Newton interpolation.

$p_1 = y_1 \quad \leftarrow$ interpolates (t_1, y_1)

$p_2 = p_1 + c(t-t_1) \leftarrow$ interpolates (t_1, y_1) & (t_2, y_2)

$p_2(t_1) = p_1(t_1) = y_1$ automatic

$$p_2(t_2) = \underset{\overset{\shortparallel}{y_1}}{p_1(t_2)} + c(t_2 - t_1) = y_2$$

$$\Rightarrow c = \frac{y_2 - y_1}{t_2 - t_1}$$

$$p_2(t) = y_1 + \frac{y_2 - y_1}{t_2 - t_1}(t - t_1)$$

$$p_2(t) = y_1 + \frac{y_2 - y_1}{t_2 - t_1}(t - t_1)$$

$$p_3(t) = p_2(t) + c(t-t_1)(t-t_2) \leftarrow \text{if we add } (t_3, y_3))$$

$$p_{n-1}(t) = y_1 + y_2\underline{\underline{(t-t_1)}} + \dots + y_n\underline{\underline{(t-t_1)}}\dots(t-t_{n-1})$$

$$= y_1 + (t-t_1)(y_2 + (t-t_2)\cdot(y_3 + \dots \quad)))$$

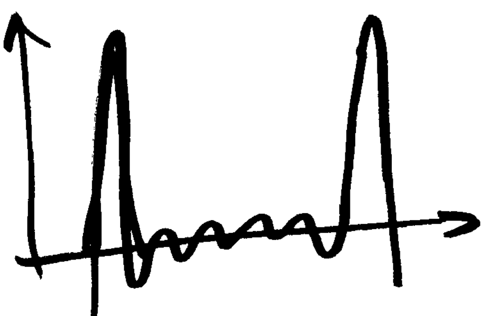
$$A = \begin{bmatrix} 1 & 0 & & 0 \\ \vdots & t_2 - t_1 & 0 & \\ 1 & t_n - t_1 & & (t_n - t_1)\dots(t_n - t_{n-1}) \end{bmatrix}$$

$Ax = y$ $\quad O(n^2)$ operations

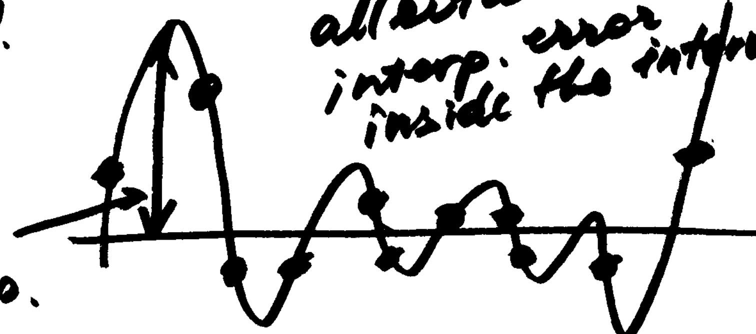
$\Rightarrow$ best of both worlds.

4) Orthogonal polynomials

$$\langle p_i, q_j \rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

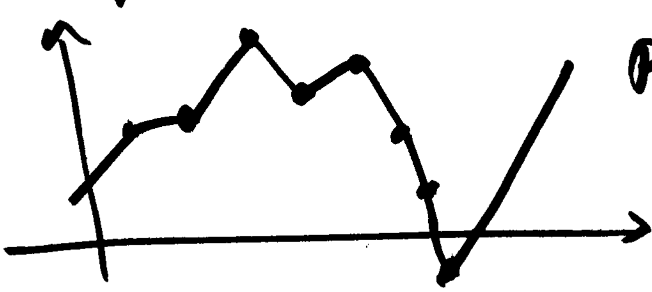


interp. error



choice of Chebyshev nodes alleviates the interp. error inside the interval

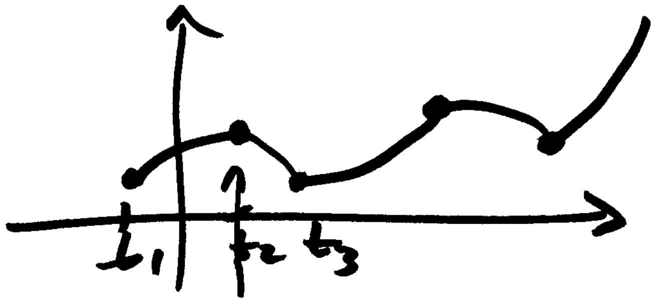
It is hard to eliminate the wiggling problem when using poly interpolation



piecewise linear is an answer

Piecewise interpolants

1) Hermite polys
2) Cubic splines
3) B-splines



Smoothness condition has to be imposed.

$S_+'(x_i) = S_-'(x_i)$

in addition, $S_+''(x_i) = S_-''(x_i)$ ← continuity of S''.

$$S(x) = \begin{cases} \underline{b_0^1} + \underline{b_1^1} x + \underline{b_2^1} x^2 + \underline{b_3^1} x^3 & \leftarrow [x_1, x_2] \\ \vdots & \\ \underline{b_0^{n-1}} + \underline{b_1^{n-1}} x + \underline{b_2^{n-1}} x^2 + \underline{b_3^{n-1}} x^3 & [x_{n-1}, x_n] \end{cases}$$

Hermite: $4(n-1)$ coefficients (unknown)

$2(n-1)$ interp. cond.

$+\; n-2$ ← cont. of 1st deriv.

$3n-4$ equations

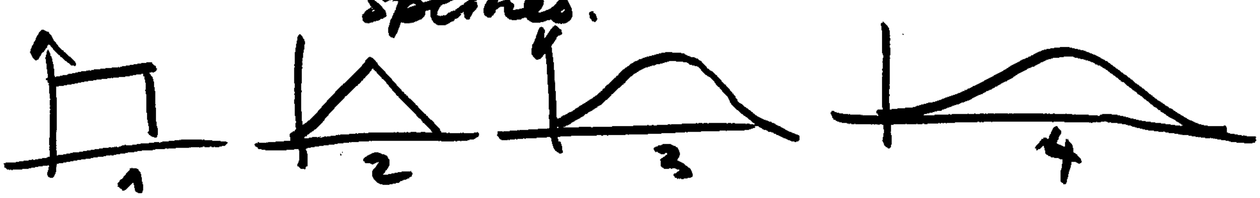
→ 4 free variables.

Spline: 2 free variables

$n-2$ addit. constraints on S''

B-spline: basis functions in the space of splines.

order



§4. Eigenvalues & Eigenvectors.

$\boxed{A\vec{x} = \lambda\vec{x}}$ $\vec{x} \neq 0$ Finding directions in which A acts by simple stretching (or expansion) or shrinking.

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \underset{\substack{\uparrow \\ x}}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \underset{\substack{\uparrow \\ 2\cdot x}}{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}$$

$\lambda(A) = \{\lambda_1, \ldots \lambda_n\}$ – spectrum

$\lambda_i = \lambda_j$ ← repeating eigenvalues can occur

$\downarrow \quad \downarrow$

$v_i \quad v_j$ ← if v_i, v_j are lin. dependent A is called defective.

$\rho(A)$ = spectral radius (used in convergence proof).

$$\left.\begin{aligned} \lambda_i &= a + ib \\ \lambda_j &= a - ib \end{aligned}\right)$$

How do we find $\lambda_1, \ldots \lambda_n$?

$(A - \lambda I)\vec{x} = 0 \qquad \underbrace{\det(A - \lambda I) = 0}_{\text{characteristic eqn.}}$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0$$

$\lambda_{1,2} = 1$.

Char. eqn. $\underbrace{P_A(\lambda)}_{\text{poly of deg} = n} = 0$ with roots λ_i.

⇒ always have n ~~di~~ roots.

$\lambda_1, \lambda_2, \lambda_3 : 1, 1, 2$

$\rightarrow$ Having A, can find $p_A(\lambda) = 0$ s.t. roots are A's eigenvalues $\lambda(A)$.

$\leftarrow$ Having a polynomial, $p(\lambda) = c_0 + c_1\lambda + \ldots + c_{n-1}\lambda^{n-1} + \lambda^n$
can form a matrix C (companion matrix) with eigenvalues = roots of $p(\lambda)$.

Polynomials do not provide an answer to the problem, especially in finite precision.

For eigenvalue problem to be nice:
$\rightarrow$ simple eigenvalues, no "defect" in eigenvector "eigenvectors span the corresp eigenspaces"

$\rightarrow$ good to have

$$\underbrace{X^{-1}AX}_{\text{Similarity transform}} = \underbrace{\begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix}}_{D\text{ - diagonal}} \rightarrow A \text{ is diagonalizable.}$$

$|X| \neq 0.$

$2 \cdot (5) \cdot \frac{1}{2} = 5$

$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$0 \cdot 5 \Rightarrow 0$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \underbrace{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}_{\bar{x}} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} = 2 \cdot \bar{x}$$

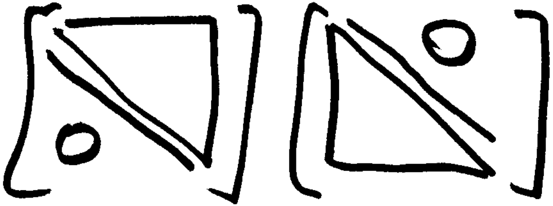
eigenvector $= \bar{x}$
$\Rightarrow 2\bar{x}$ is also an eigenvector

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\bar{x}} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} = 3\bar{x}$$

$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ spans S_{v_1}

$v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ Spans S_{v_2}

Types Hessenberg



Conditioning is affected by transforms

$$\underline{X^{-1}AX}$$
similar