Math 685

Lecture 6

Project 2:

save ← save as a file variable/workspace

a = load('file.txt')
dat

$$p = \beta_i + \beta_1 x + \beta_2 x^2 \qquad\qquad p = \beta_1 + \beta_2 e^x + \beta_3 x$$

$$A = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & \vdots \\ 1 & \vdots & \\ 1 & x_n & x_n^2 \end{pmatrix} \qquad\qquad A = \begin{pmatrix} 1 & e^{x_1} & x_1 \\ 1 & e^{x_2} & \vdots \\ 1 & \vdots & \vdots \\ 1 & e^{x_n} & x_n \end{pmatrix}$$

Test your code for:

$$\cancel{A =} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} \simeq \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{b}}$$

(a) Find exact solution for linear least squares in terms of ε

(b) Analyze behavior for different ε, e.g. $\varepsilon \sim \sqrt{\varepsilon_{mach}}$, $\varepsilon \sim \varepsilon_{mach}$

Symbolic toolbox:

syms s

Methods for least squares:

1) $A^TAx = A^Tb$ ← fastest

Normal eqns

"→" Conditiong

"−" non-accurate result

2) Augmented system

$r = b - Ax \rightarrow \begin{cases} r + Ax = b \\ A^Tr = 0 \end{cases}$

augm. vector $\begin{pmatrix} r \\ x \end{pmatrix}$

$$\begin{pmatrix} I & A \\ A^T & 0 \end{pmatrix}\begin{pmatrix} r \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

"+" more robust, not as sensitive

"−" complexity ← storage of A

3) Transformations (P)

have to preserve the norm $\|P(Ax-b)\|_2 = \|Ax-b\|_2 = \|r\|_2$

⇒ orthogonal matrices do the job.

$$Q^T = Q^{-1}, \quad Q^TQ = I$$

$$A = QR, \quad Q\text{-orthogonal}$$

$$\|r\|_2^2 = \|b - Ax\|_2^2 = \|b - QRx\|_2^2 =$$

$$\|\underbrace{Q^Tb}_{c} - Rx\|^2 = \|c - Rx\|^2$$

Any orthogonal Q will do,

but there are some choices.

$$Q = [Q_1 \; Q_2]$$

$$A = Q\begin{bmatrix} R \\ 0 \end{bmatrix} = [Q_1 \, Q_2]\begin{bmatrix} R \\ 0 \end{bmatrix} = Q_1 R$$

reduced QR

rank(A)<n, for rank-deficient problems, R will be singular.

Column-pivoted QR can still solve $Rx=c$, even in rank-deficient.

<u>Methods for finding Q</u>:

P projector onto span(A).

Recall $P = A(A^TA)^{-1}A^T$ $\boxed{Ax = Pb}$

$P^2 = P$ (idempotent operator

suffers from instability

In orthogonal case, $Q^TQ = I$

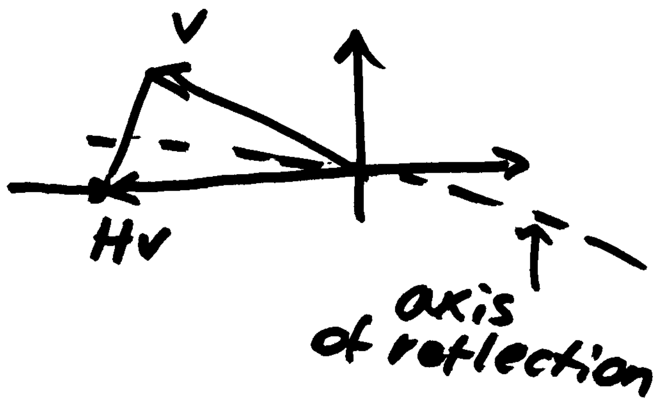
(a) Householder

(b) Givens

(c) Gram-Schmidt

$$H = I - 2\frac{v \cdot v^T}{v^Tv}$$

$$\|Hv\| = \|v\|$$

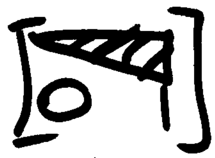


Householder ~~pr~~ reflects in $\mathrm{span}(A)^\perp$ so that all or some of vector components become zero, except for pivoting positions.

$$a=\begin{bmatrix}1\\1\\1\end{bmatrix} \xrightarrow{H} \begin{bmatrix}1\\0\\0\end{bmatrix}$$

1st column of A

$$\begin{bmatrix} A \end{bmatrix} \rightsquigarrow$$



$$Ha=\begin{pmatrix}\alpha\\0\\ \vdots\\0\end{pmatrix}=\alpha e_1$$

1st column of $I = \begin{pmatrix}1&0\\0&1\end{pmatrix}$

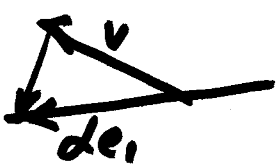
$$\left(I-2\frac{v\cdot v^T}{v^Tv}\right)\vec{a}=\alpha e_1$$

$$\vec{a}-2\vec{v}\cdot\underbrace{\left(\frac{v^Ta}{v^Tv}\right)}_{\text{const}}=\alpha e_1 \quad \Big| \times\left(\frac{v^Tv}{v^Ta}\right)$$

$$\left[\frac{v^Tv}{2v^Ta}\right](\vec{a}-\alpha\vec{e}_1)=\vec{v} \qquad H=I-2\frac{v\cdot v^T}{v^Tv}$$

$$\vec{v}=\vec{a}-\alpha\vec{e}_1 \qquad v=\underline{c}\cdot u \rightarrow c \text{ will cancel}$$

$$\|\vec{v}\|_2=\|\alpha e_1\|_2 \Leftrightarrow |\alpha|$$



$$\alpha = -\mathrm{sign}(a_1)\cdot\|a\|_2$$

$$\|v_{start}\|_2=\|\text{result}\|_2$$

$$v=\begin{pmatrix}2\\1\\2\end{pmatrix} \text{ onto } \mathrm{span}\left\{\begin{pmatrix}2\\1\\0\end{pmatrix}\right\} \rightsquigarrow \left\|\begin{pmatrix}2\\1\\2\end{pmatrix}\right\|_2=\left\|\alpha\begin{pmatrix}2\\1\\0\end{pmatrix}\right\|_2$$

$$v = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \qquad H v = \alpha e_1 \Longrightarrow \|v\|_2 = \|\alpha e_1\|_2 = |\alpha|$$

output

$$\|v\|_2 = 3 \rightarrow \alpha = -\mathrm{sign}(v_1) \cdot 3 = -3$$

$$\mathfrak{v} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}$$

$$H = I - 2\frac{\mathfrak{v} \cdot \mathfrak{v}^T}{\mathfrak{v}^T \mathfrak{v}} \qquad \|H\mathfrak{v}\|_2 = \|\mathfrak{v}\|_2$$

(b) <u>Givens rotations</u>:

$$\begin{pmatrix} c & s \\ -s & c \end{pmatrix} \qquad c = \frac{a_1}{\sqrt{a_1^2 + a_2^2}} = \cos\alpha$$

$$s = \frac{a_2}{\sqrt{a_1^2 + a_2^2}} = \sin\alpha$$

$$\tan\alpha = \frac{s}{c} = \frac{a_2}{a_1}$$

$$v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{G} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \cancel{G=} \qquad G = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(1) rotate 1 & 2 $\quad G_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \rightarrow \begin{pmatrix} d_1 \\ \cdot \\ \cdot \end{pmatrix}$

(2) rotate 1 & 3 $\quad G_2 = G_1 \quad \begin{pmatrix} d_2 \\ 0 \\ 0 \end{pmatrix}$

$$\|G_2 G_1 v\|_2 = \|v\|_2$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$m \times n$, rank$(A) < n \Rightarrow$ deficiency

$m > n$

1 sol.

coinciding planes
inf. many

$$\begin{array}{c} x\ y \\ \begin{bmatrix} & \\ & \end{bmatrix} \end{array} \qquad y = 2x$$

(c)
<u>SVD</u> $A = U\Sigma V^T, \quad \Sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{pmatrix}$ σ_i - singular values

$$\text{cond}(A) = \frac{\max \sigma_i}{\min \sigma_i}$$

for least squares : $A^TAx = A^Tb$

$A = U\Sigma V^T$, U - orth. basis for ~~columns~~ rows
V - orth. basis for columns

$$U = [U_1 \ U_2] \quad V = [V_1 \ V_2] \quad \Sigma = \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix}$$

$$A = [U_1 \ U_2]\begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix} V^T = U_1 \Sigma_1 V^T \leftarrow \text{reduced SVD}$$

$$A^T = V\Sigma_1^T U_1^T$$

$$A^TA = V\Sigma_1^T \underbrace{U_1^T \cdot U_1}_{I} \Sigma_1 V^T = V \underbrace{\Sigma_1^T \Sigma_1}_{\cancel{\Sigma_1}\Sigma_1^2 V^T} V^T = \cancel{\Sigma V}^T$$

$$A^TAx = A^Tb$$

$$\underbrace{V\Sigma_1^T}\Sigma_1 V^T x = \underbrace{V\Sigma_1^T}_{V^{-T} = V} U_1^T b \Rightarrow \Sigma_1 V^T x = U_1^T b$$

$$\boxed{x = V\Sigma_1^{-1} U_1^T b}$$

SVD: $x = V\Sigma_1^{-1}U_1^T b$

$$\Sigma_1^{-1} = \begin{pmatrix} \frac{1}{\sigma_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sigma_n} \end{pmatrix}$$

$$\boxed{x = \sum_{\sigma_i \neq 0} \frac{u_i^T b}{\sigma_i} v_i}$$

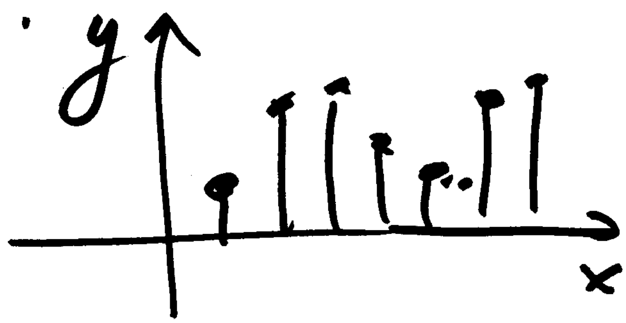
$$A = U\Sigma V^T = \sigma_1 E_1 + \dots + \sigma_n E_n$$

$$E_i = u_i v_i^T$$

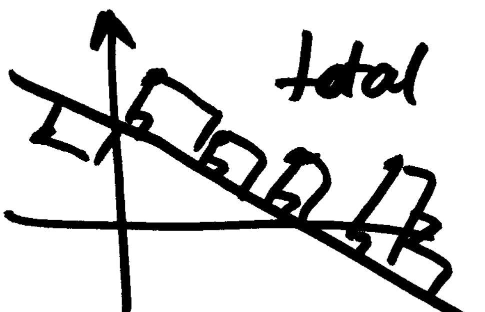
$\sigma_k, \dots \sigma_n \approx 0$, $k > 1 \Rightarrow A = \sigma_1 E_1 + \dots + \sigma_{k-1} E_{k-1}$

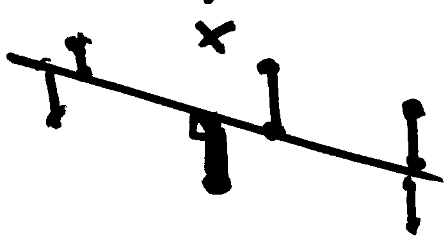
* Preferred technique for rank deficient systems.

Total least squares

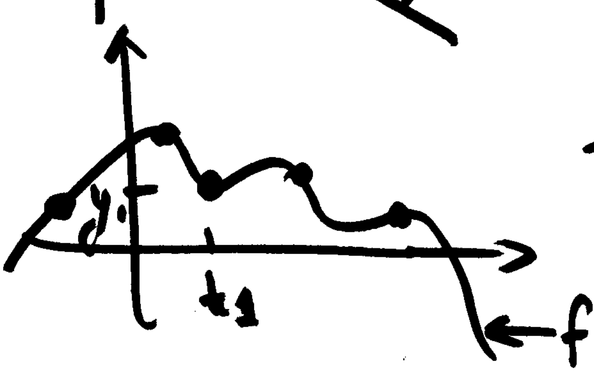


$\bar{A} = [A \ \ b]$





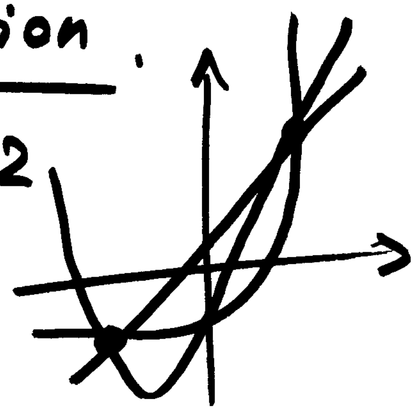
$f(t_i) = y_i$



Interpolation

Polynomial

n pts $\quad n=2$

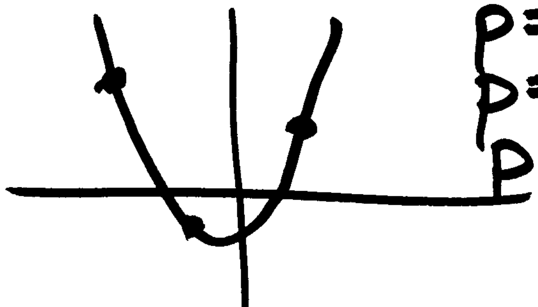


$p = \deg(f)$

$p=1 \rightarrow$ unique

$p>1 \rightarrow$ inf. many interpolants

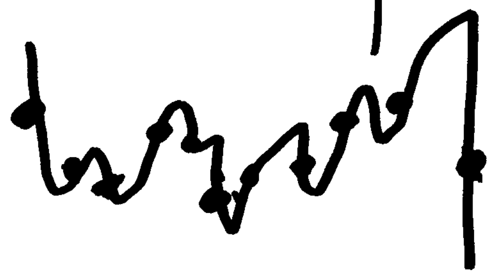
$n=3$



$p=1 \leftarrow$ cannot fit this data

$p=2 \leftarrow$ unique

$p>2 \leftarrow$ inf. many interp.



deg(poly) large, error increases due to high derivative

For interpolants:

1) Simplicity
2) Basis functions are important

$$y_i = p(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n$$

$$\text{monomials} \longrightarrow \quad 1 \quad x \quad x^2 \quad \dots \quad x^n$$

$$\begin{bmatrix} 1 & x_1 & x_1^n \\ \vdots & & \\ 1 & x_n & x_n^n \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_n \end{bmatrix} = \begin{bmatrix} y_i \end{bmatrix}$$

$$A \quad c \quad = y$$

$c = (c_0, \dots c_n)$ unknowns