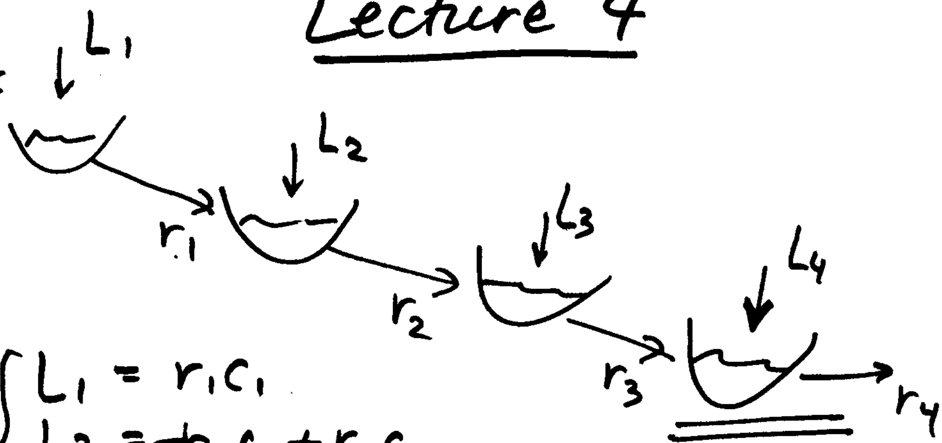
Math 685
Lecture 4

Project 3



$$\begin{cases} L_1 = r_1 c_1 \\ L_2 = -r_1 c_1 + r_2 c_2 \\ L_3 = -r_2 c_2 + r_3 c_3 \\ L_4 = -r_3 c_3 + r_4 c_4 \end{cases}$$

Remark: Forward error: $\underbrace{\frac{\|\Delta x\|}{\|\tilde{x}\|}}_{\text{accuracy}} \le \underbrace{cond(A)}_{\text{sensitivity}} \cdot \frac{\|r\|}{\|A\| \cdot \|\tilde{x}\|}$ stability

Backward error: $\frac{\|r\|}{\|A\| \cdot \|\tilde{x}\|} \le \frac{\|E\|}{\|A\|}$ $\qquad (A+E)\tilde{x} = b$

$\|r\|$ - large $\Rightarrow$ big $\|E\|$.

HW3.

① $cond(A) = \|A\| \cdot \|A^{-1}\|$ $\qquad A^{-1} = \begin{pmatrix} \frac{1}{4} & & 0 \\ & \frac{1}{6} & \\ 0 & & \frac{1}{2} \end{pmatrix}$

$$\frac{\|\Delta x\|}{\|\tilde{x}\|} \le \underline{cond(A)} \cdot \frac{\|\Delta b\|}{\|\tilde{b}\|}$$

$\|A\|_1 = 6 \quad \|A^{-1}\|_1 = \frac{1}{2} \qquad cond_1(A) = 3$

② $Ax = b$

$DAx = Db \qquad |D| \neq 0.$ Scaling.

diag.

$DAx = Db$

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $D = \begin{pmatrix} 10^4 & 0 \\ 0 & 1 \end{pmatrix}$ $cond(A) = 1$
$Cond(DA) = 10^4$

$DA = \begin{pmatrix} 10^4 & 0 \\ 0 & 1 \end{pmatrix}$

$A = \begin{pmatrix} \boxed{20} & 0 \\ 5 & 1 \end{pmatrix}$ $D = \begin{pmatrix} 10^{-4} & 0 \\ 0 & 1 \end{pmatrix}$ $DA = \begin{pmatrix} \underline{20 \cdot 10^{-4}} & 0 \\ \underline{5} & 1 \end{pmatrix}$

$$\begin{pmatrix} 10^{-4} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 20 & 0 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 20 \cdot 10^{-4} & 0 \\ 5 & 1 \end{pmatrix} \quad \begin{pmatrix} \varepsilon & 0 \\ 1 & 1 \end{pmatrix}$$

③ $A = LU$ $|A| \neq 0.$

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $PA = LU$ exists
$A \neq LU$ zero pivot

Q: if $A = LU$ exists $\Rightarrow |A| \neq 0$?

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

④ $r = Ax - b$

r-small $\overset{?}{\Rightarrow}$ $|x - \tilde{x}|$-small <u>NO.</u>

r-large $\overset{?}{\Rightarrow}$ $|x - \tilde{x}|$-large <u>NO</u>.
but: <u>YES</u>

$\frac{\|E\|}{\|A\|} \geq \frac{\|r\|}{\|A\| \cdot \|\tilde{x}\|}$ – does mean that backward error is large if r is large.

(5) $Ax = b$ 12-digit precision

Estimated loss of accuracy $\approx \log_{10} \text{cond}(A)$

$\log_{10} \text{cond}(A) = 12 \Rightarrow \text{cond}(A) \sim 10^{12}$

$(\#\text{digits in precision}) - \log_{10} \text{cond}(A) =$

$=$ # of sign. digits that are accurate in your answer

(6) $(A - uv^T)^{-1} = A^{-1} + A^{-1}u(1 - v^T A^{-1} u)^{-1} v^T A^{-1}$

1) $\underline{Ax = b} \rightsquigarrow A^{-1}$ by LU decomp.

2) $B = A - uv^T \leftarrow$ rank-1 update

To compute $By = b$: $B = A - uv^T$

$y = (A - uv^T)^{-1} b$

SM formula: $y = \underbrace{A^{-1}b}_{x} + \underbrace{A^{-1}u}_{z} \underbrace{(1 - v^T \underbrace{A^{-1}u}_{z})^{-1} v^T \underbrace{A^{-1}b}_{x}}_{\text{const}}$

$u = \begin{bmatrix} \\ \\ \end{bmatrix}$ $v^T = [\cdot \; \cdot \; \cdot \; \cdot]$ $v^T z, \; v^T x$

$(1 \times n)(n \times 1) \to 1 \times 1$

$\Rightarrow y = x + (\text{const}) \cdot z$

$$(A - uv^T)^{-1} = A^{-1} + A^{-1}u(1 - v^T A^{-1} u)^{-1} v^T A^{-1}$$

$$\cancel{I} = (A - uv^T)A^{-1} + (A - uv^T)A^{-1}u(1 - v^T A^{-1} u)^{-1} v^T A^{-1}$$

$$\cancel{I} - uv^T A^{-1}$$

$$\underline{uv^T A^{-1}} = (A - uv^T)A^{-1}u(1 - v^T A^{-1} u)^{-1} \underline{v^T A^{-1}}$$

$$u = (A - uv^T)A^{-1}u(\underline{1 - v^T A^{-1} u})^{-1}$$

$$(1 - v^T A^{-1} u)u = (A - uv^T)A^{-1}u$$

$$= (I - \underbrace{uv^T A^{-1}})u = u -$$

$$v^T A^{-1} u = \text{const} = c$$

$$\underline{(1-c)u = u - uc} \qquad cu = uc$$

identity

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(6) $[L, U, \underline{P}] = lu(A)$ ← MATLAB LU decomp.

permutation

$$Ax = b \qquad PA = LU$$

$O(n^3)$ effort

$$PAx = Pb$$

$$\underbrace{LUx}_{z} = Pb$$

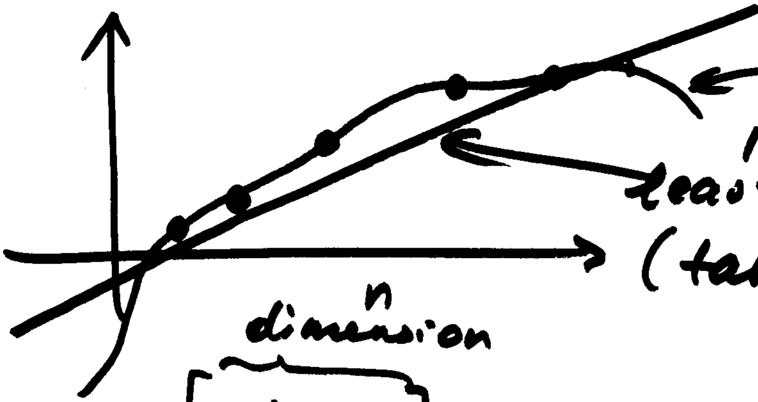
$$\begin{cases} Lz = Pb & \leftarrow \text{1st step} \\ Ux = z & \leftarrow \text{2nd step} \end{cases}$$

order $O(n^2)$ effort

Sherman-Morrison

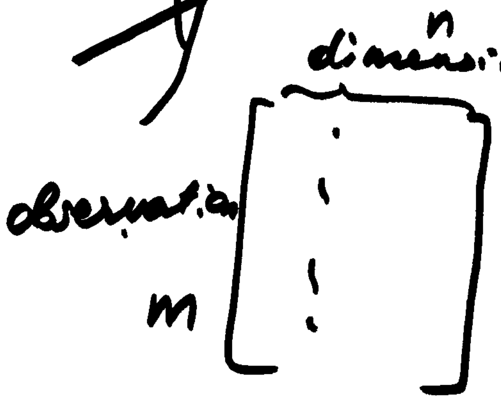
Complexity: $O(n^2)$ vs. $O(n^3)$ for GE.

Least squares.



$p(x)$ - interpolant

linear least squares approx. (taking into account error in dataset)



$m \times n$

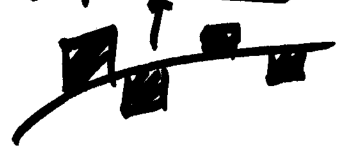
$m > n \to$ overdetermined

$m < n \to$ underdetermined

will have no solution in general sense

$Ax \cong b \to$ looking for best x that is close enough to solution.

$$\min_x \|b - Ax\|_2 = \min_x \left(\sum_{i=1}^{n} (b - Ax_i)^2 \right)^{1/2}$$



$$\sum (y_i - f(t_i, x))^2 \longrightarrow \min$$

$$f = \underline{a} + \underline{b}t + \underline{c}t^2 \leftarrow (t_i, y_i) = i = 1 \ldots 5$$

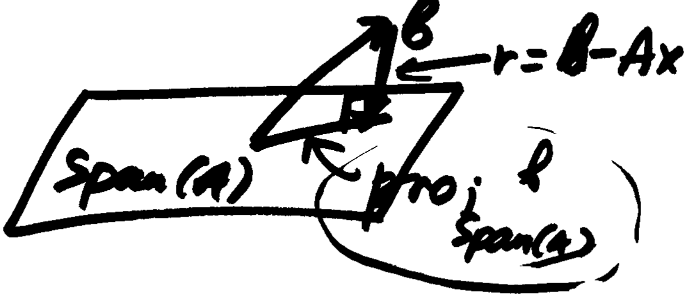
$$\begin{cases} a + bt_1 + ct_1^2 = y_1 \\ a + bt_2 + ct_2^2 = y_2 \\ \quad \cdot \quad \cdot \quad \cdot \\ a + bt_5 + ct_5^2 = y_5 \end{cases} \to \begin{array}{ccc} a & b & c \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & t_1 & t_1^2 & y_1 \\ 1 & t_2 & t_2^2 & y_2 \\ \vdots & & & \vdots \\ 1 & t_5 & t_5^2 & y_5 \end{array}\right]$$

Least squares solution:

$$\boxed{A^T A x = A^T b} \leftarrow \text{normal equations}$$

$$Ax = b$$



Pseudoinverse: $A^+ = (A^T A)^{-1} A^T$

$\text{cond}(A) = \|A\| \cdot \|A^+\|$

$m \times n,\ m > n$

$\underline{x = A^+ b}$

least squares solution

$$A^T A x = A^T b$$

$$x = (\underbrace{A^T A}_{SPD})^{-1} A^T b \leftarrow \text{normal eqn.}$$

$$(A + E)x \cong b$$

$$\frac{\|\Delta x\|}{\|x\|} \lessapprox \left(\text{cond}^2(A) \cdot \tan(\theta) + \text{cond}(A)\right) \cdot \frac{\|E\|}{\|A\|}$$

$\beta = 3$

Bandwidth

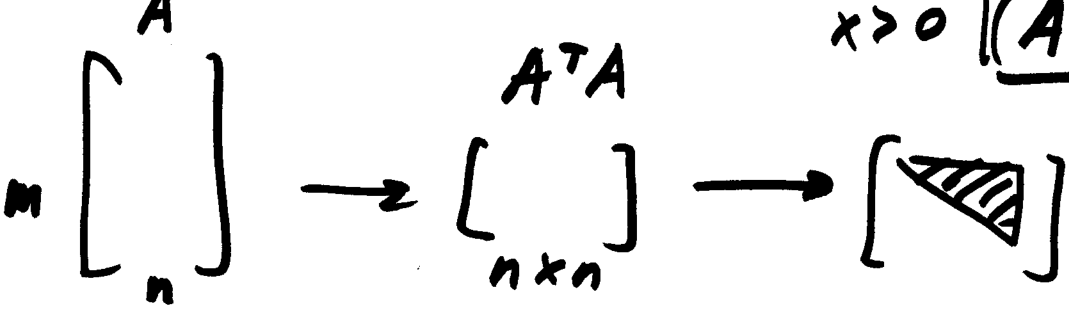
$O(\beta^2 n)$ - inversion

$O(\beta n)$ - storage

"sparse" in MATLAB.

$$\boxed{A^T = A}$$

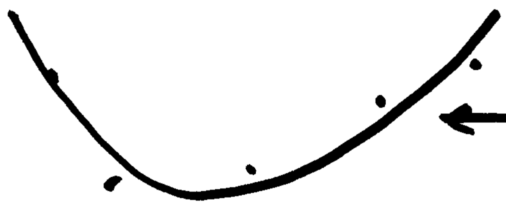
$$x > 0 \quad \boxed{(Ax, x) > 0}$$



$A^TA = LL^T$ $\qquad \boxed{A^TAx = A^Tb}$

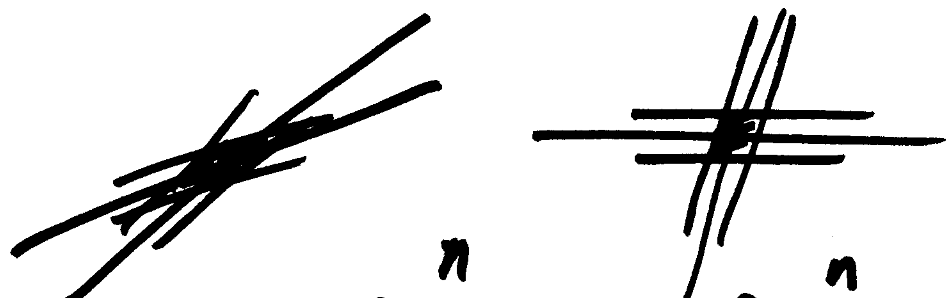
$$L\underbrace{L^Tx}_{z} = A^Tb$$

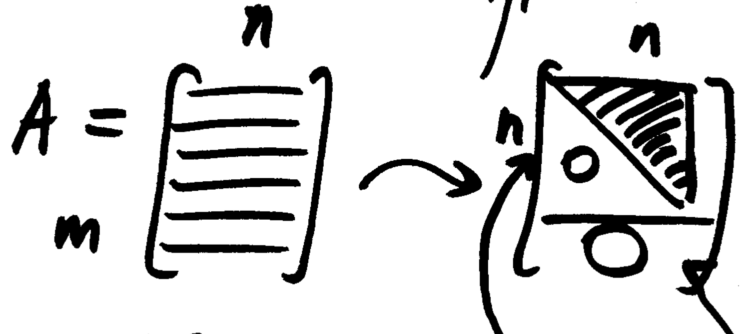
2 steps:
1) $Lz = A^Tb$
2) $L^Tx = z$



$p(x) = a + bt + ct^2$

$\vec{x} = [0.086 \quad 0.4 \quad 1.429]^T$





$\boxed{A = Q_1R}$

$Q\begin{bmatrix}R\\0\end{bmatrix}$ $\qquad Q = [\underbrace{Q_1}_{n} \ \underbrace{Q_2}_{m-n}]$

$$Q_1^TAx = \boxed{Rx = c_1} = Q_1^Tb$$

LU - by elimination matrices

QR - by orthogonal matrices

$Q_n \ldots Q_2Q_1A \longrightarrow$