MATH 685 (02/04)

1. Function handling in Matlab:

   (a) @f - function handler

   out = test(@f, x_0)

   (b) f = inline('x^2-1');

   out = test(f);

2. Ex. 1

   (a) 0:h:1 ← mesh size $h = \frac{1-0}{40}$ (0:40)/40

   (b)

   (c) y=sin(x) ← values of sin at grid points

Ex. 2

$n! = 1\cdot 2\cdot 3\cdot \ldots \cdot n$

Ex. 3.

$$f'(x) \approx \frac{f(x+h)-f(x)}{h}$$

$$h = 10^{-1}, 10^{-2}, 10^{-3}, \ldots 10^{-15} \to 0$$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$$

$$f(x+h) = f(x) + \cancel{h} f'(x) + \frac{h^2}{2} f''(x) + \ldots$$
$$- f(x)$$

forward difference

$$\frac{f(x+h)-f(x)}{h} = f'(x) + \boxed{\frac{h}{2} f''(x) + \ldots}$$

$O(h)$ truncation $O(h)$

$f = \sin(x)$
$f' = \cos(x)$
$f'' = -\sin(x)$

$$\to \frac{f(x+h)-f(x-h)}{2h} \text{ instead}$$

central difference

$$\Longrightarrow O(h^2)$$

If $0 < h < 1 \Rightarrow h^2 \ll h$

$O(h^2)$ approximates better than $O(h)$ above.

$$\underbrace{d_0.d_1 d_2 \ldots d_{p-1}}_{\text{mantissa}} \cdot \beta^{E}$$

β ← base

E ← exponent $\boxed{L \le E \le U}$

$\beta, E, p \rightarrow$ machine dependent

if actual $E < L \Rightarrow$ underflow

$E > U \Rightarrow$ overflow

$0.534 \cdot 10^6$

$\rightarrow \underline{5.34 \cdot 10^5}$ ← normalized

0,1 ← binary repres.

$\boxed{1}.01100\ldots$

$\frac{1}{3} = 0.333\ldots$ ← not repr. exactly in decimal arithmetic

$\underline{fl\left(\frac{1}{3}\right)} \neq \frac{1}{3}$

approximation

$5.6 \sim 6$

$5.4 \sim 5$

$6.5 \sim 6 \quad 5.5 \sim 6$

$\boxed{\varepsilon_{mach}}$: the smallest number s.t.

$1 + \varepsilon_{mach} > 1$

if $\varepsilon < \varepsilon_{mach}$, $1 + \varepsilon_{mach} = 1$

$\varepsilon_{mach} \neq 0$ in machine

$\varepsilon_{mach} \neq N_{underflow}$

$$x = \underbrace{1.00\ldots0}_{p}(1) \sim \underbrace{1.\overbrace{0\ldots01}^{p-1}}_{p}$$

$$\cancel{\text{error}}\quad \varepsilon_{mach} = \begin{cases} 2^{1-p} \leftarrow \text{chopped} \\ \frac{1}{2}\cdot 2^{1-p} \leftarrow \text{rounded} \end{cases}$$

$$\text{error: } \left|\frac{fl(x) - x}{x}\right| \leq \varepsilon_{mach}$$

Ex.

$$\begin{array}{r} 0.1\cdot 10^1 \\ -\ 0.99\cdot 10^0 \\ \hline \end{array}$$

exact $= 1 - 0.99$
$= \underline{0.01}$

If $p = 3$, $\beta = 10$

machine arithm. chopped

$$\begin{array}{r} 0.10\cdot 10^1 \\ -\ 0.09\cdot 10^1 \\ \hline 0.01\cdot 10^1 \end{array} = \underline{0.1}$$

10 times larger

Loss of significance:

$$\begin{array}{r} 0.123456 \\ -\ 0.123452 \\ \hline 0.000004 \end{array} \rightarrow 4.0 \cdot 10^{-6} \rightarrow$$ 1 sign digit

← 6 sign. digits

$$\overset{\uparrow}{1}.\underbrace{24005}_{5}000$$ ← 6 sign. digits

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$\varepsilon < \varepsilon_{mach}$

$$(1+\varepsilon) - (1-\varepsilon) = 2\varepsilon$$

$$\downarrow \qquad\qquad \downarrow$$

$$1 - 1 = 0$$

Ex. 1) $\sin x - x$, $x \sim 0$

2) $\sin(x+h) - \sin x$, $h \sim 0$.

$$\sin x \approx x - \frac{x^3}{3} + \frac{x^5}{5!}$$

$$\sin x - x \approx -\frac{x^3}{3} + \frac{x^5}{5!} + \dots$$

$$\frac{1-\sqrt{x+1}}{x} = \frac{1+\sqrt{x+1}}{1+\sqrt{x+1}} \cdot \frac{1-\sqrt{x+1}}{x} = \frac{\cancel{1} - x - \cancel{1}}{x\left(1+\sqrt{x+1}\right)}$$

$$x = \frac{-b+\sqrt{b^2-4ac}}{2a}$$

b - large

$ac \ll b$.

$$-b+\sqrt{b^2-\underbrace{\varepsilon}}$$

$\overset{??}{b}$

loss of Significance.

$$ax^2+bx+c=0$$

$$= \frac{-b+\sqrt{b^2-4ac}}{2a} \cdot \frac{+b+\sqrt{b^2-4ac}}{+b+\sqrt{b^2-4ac}}$$

Fact: $b^2-a^2=(b-a)(b+a)$

$$\frac{\cancel{b^2}-(\cancel{b^2}-4ac)}{2a\left(b+\sqrt{b^2-4ac}\right)} = \frac{4ac}{2a(\cdot\cdot)}$$

$$= \boxed{\frac{2c}{b+\sqrt{b^2-4ac}}}$$

$a, b, c \ll 1$ $\quad a x^2 + b x + c = 0$

small

$$0 = \underline{10^{-6}}x^2 + \underline{3\cdot 10^{-6}}x + \underline{10^{-6}}$$

$$0 = x^2+3x+1$$

1) $\underline{\frac{1}{1-x} - \frac{1}{1+x}}$ vs. $\frac{1+x-(1-x)}{1-x^2}$

2) x^2-y^2 vs. $(x-y)(x+y)$

Chapter 2.

$\boxed{Ax=b}$

$$\begin{cases} x_1+2x_2=1 \\ 2x_1+x_2=2 \end{cases} \Rightarrow A=\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$\vec{x}=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\nwarrow$ $b=\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\det A = 1\cdot 1 - 2\cdot 2$

Singular system: if $\det A = 0$

$$\begin{cases} x_1+2x_2=1 \\ 2x_1+4x_2=2 \end{cases} \quad A=\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$\operatorname{Ker} A = \{x \mid Ax=0\}$

kernel

$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \xrightarrow{\text{ERO}} \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}$

2x

② − 2·①
row   row

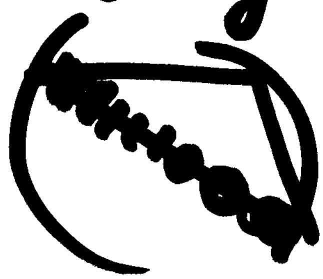
$\begin{pmatrix} x & x & x \\ x & x & x \\ x & x & x \end{pmatrix} \xrightarrow{\text{ERO}}$

rank $A = n$

A − $n \times n$ square matrix

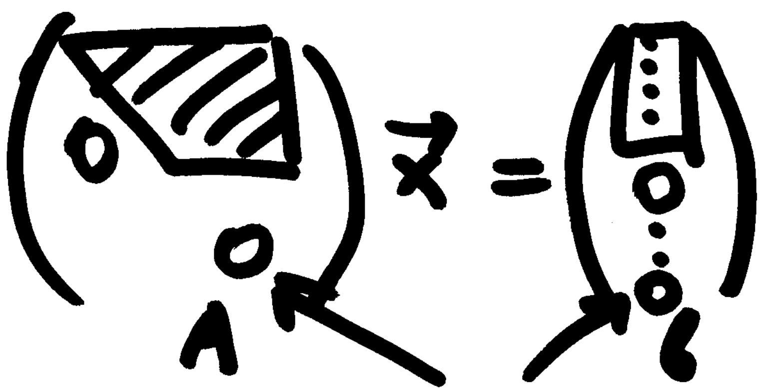
if rank $A = n$ → no zeros on diagonal

if rank $A < n$ ⇒



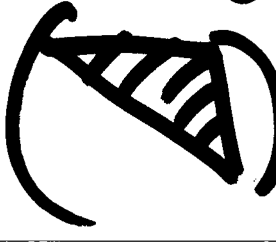
$A =$

$Ax = b$



$a \cdot x = b$
$a = 0$, $b = 0$

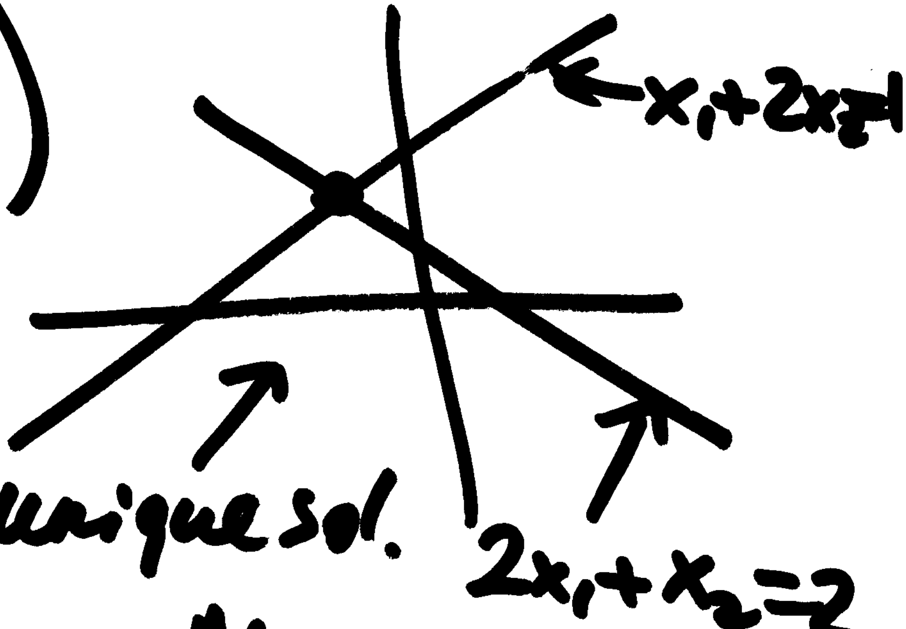
$0 \cdot x_n = b_n$



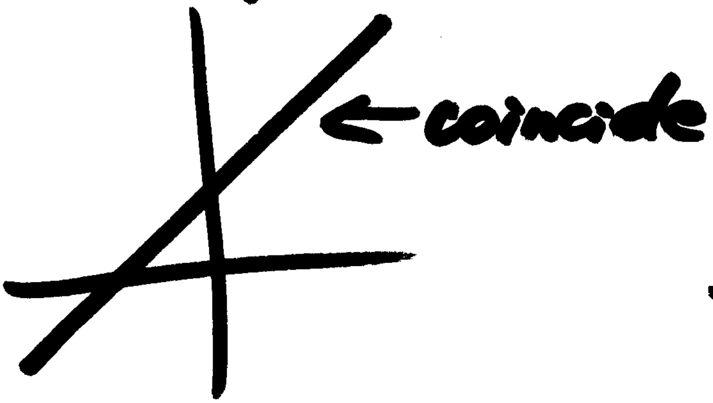
$\vec{x} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

$\mathbb{R}^n = Ker(A) \oplus Span(A)$

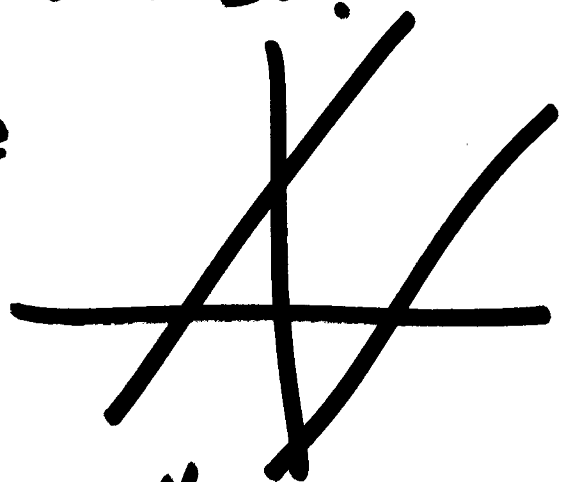
$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 1 & 2 \end{array}\right)$$



Multiple sol.



No sol.



if # unknowns = # equations,

$\oplus$ System is nonsingular

$\Longrightarrow$ unique solution.

Vector norms

$$\vec{x} = (x_1, \ldots x_n)$$

$$|x| = \sqrt{x_1^2 + \ldots + x_n^2} \leftarrow \text{length}$$

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} - \text{p-norm}$$

$p = 2 \rightarrow$ length, Euclidean distance

$$p = 1 \rightarrow \|x\|_1 = \sum |x_i|$$

$$\|x\|_\infty = \max_{1 \le i \le n} |x_i|$$

All p-norms are equiv.

$$\|x\|_\infty \le \|x\|_2 \le \|x\|_1$$

$$\|x\|_1 \le \sqrt{n}\,\|x\|_2$$

$$\|x\|_{\cancel{\infty}\,2} \le \sqrt{n}\,\|x\|_\infty$$

$$\|x\|_1 \le n\,\|x\|_\infty$$

$$\|x\|_2 = 1, \quad \|x\|_1 = 1, \quad \|x\|_\infty = 1.$$

$$\|A\|_1 = \max\{\text{col. sums}\} = 6$$

$$A = \begin{array}{cl} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} & \begin{matrix} \longrightarrow 3 \\ \longrightarrow 7 \end{matrix} \\ \begin{matrix} \downarrow & \downarrow \\ 4 & 6 \end{matrix} & \end{array}$$

$$\|A\|_\infty = \max\{\text{row sums}\} = 7$$

For numbers, $|(-2)\cdot 5| = |-2|\cdot|5|$

For matrices, $\|A\cdot B\| \neq \|A\|\cdot\|B\|$

$$\leq \|A\|\cdot\|B\|$$

$$\|A\| = \sup \frac{\|Ax\|}{\|x\|} \longleftarrow \text{max stretching}$$

$$\Rightarrow \|Ax\| \leq \|\underset{\uparrow \atop \text{matrix}}{A}\| \cdot \|\underset{\uparrow \atop \text{vector}}{x}\|$$

$$\text{cond}(A) = \|A\|\cdot\|A^{-1}\|$$

$\text{cond}(A) = 1 \rightarrow$ A is almost like identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

II cond(A) >> 1 ⇒ ill-conditioned
very sensitive

$$\underbrace{\frac{\|\Delta x\|}{\|x\|}}_{\text{relative solution error}} \le \text{cond}(A) \cdot \underbrace{\frac{\|\Delta b\|}{\|b\|}}_{\text{relative input error}}$$

$$A\vec{x} = \vec{b}$$

Δb - error for r.h.s.

$$\|b\| \sim 10^6$$